GRAPHIC AND ANALYTIC STATICS

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IN

THEORY AND COMPARISON:

THEIR

PRACTICAL APPLICATION TO THE TREATMENT OF STRESSES IN ROOFS, SOLID GIRDERS, LATTICE, BOWSTRING AND SUSPENSION BRIDGES, BRACED IRON ARCHES AND PIERS, AND OTHER FRAMEWORKS:

TO WHICH IS ADDED

A CHAPTER ON WIND PRESSURES.

BY

ROBERT HUDSON GRAHAM, C.E.

CONTAINING DIAGRAMS AND PLATES TO SCALE, WITH NUMEROUS EXAMPLES, MANY TAKEN FROM EXISTING STRUCTURES.

Specially arranged for Class-work in Colleges and Universities.

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PREFACE.

THE branch of science known as *Graphic Statics* is gradually forcing itself on the attention of mathematicians and civil engineers as a rapid and elegant means of calculation, admirably adapted for finding the stresses induced in the members of roofs, bridges, and other frame-work types of structure.

The aim of this work is, first of all, to place in a clear light the theory and relations of Graphic and Analytic Statics; and, then, to illustrate their practical application to the treatment of stress in the familiar forms of iron and wooden frameworks.

Part I. deals with the fundamental principles of Graphic Statics, in which each proposition is proved, step by step, by the sole aid of pure geometry, without consciously leaving any serious gaps in the demonstrations to distress and discourage beginners.

Part II. comprises the dual treatment of roof and bridge-structures by graphic and analytic methods; where results, formerly obtained by analytical calculation, are deduced in their analytical shape from the forms and relations existing in the graphic diagram. These new graphic demonstrations of familiar formulæ will be found to agree with those analytically determined by Rankine and Bresse in various parts of their works on Applied Mechanics.

Another special feature of the Second Part is the treatment of a given roof or bridge by two methods which mutually check and corroborate each other. The roof or bridge is first

taken truss by truss, and the reciprocal diagrams of the independent trusses are given in separate form. The resultant stress in any particular member, appertaining to several trusses, is then found by taking the *graphic sum* of the component lines in the diagrams of the independent trusses of which it forms a part. The same frame-work is subsequently treated as a whole, and the reciprocal diagram of the resultant stresses, directly determined, serves to confirm the same values found by the indirect method of division into separate trusses, and summation of the component stresses.

In another article the analytical and graphic methods of sections are explained, and compared in application to the same example. The same comparison is made for a half-lattice girder; and combined lattice forms are exhibited under two modes of treatment, being first subjected to the process of truss-division and summation already explained, and then to the general graphic method by which the resultant stresses are found in one operation.

Part III. embraces the graphic and analytic treatment of direct stress; extension under stress; couples of forces; resultants and centres of stress; centres of gravity; moments of all kinds; and finally straight beams and girders of various forms, considered both under concentrated, uniform, and moving loads, as well as in the state of equilibrium.

At the end of each chapter there is added, in the form of exercises for the student, a set of examples, most of which are original and many taken from existing structures. These examples, which have entailed much labour upon me in the preparation both of question and answer, will, I hope, not only add a distinctive feature to the book, but also materially increase its usefulness and lend an additional interest and reality to the work of the mathematical and engineering student. The results are the outcome of my own calculations repeatedly checked, and are given on my own authority.

In connexion with this part of the subject I have to acknow-

ledge much kindness on the part of Professor Main of the Royal School of Mines, at whose suggestion I undertook the preparation of examples, and by whose advice I also added the chapter on the system of lettering introduced by Bow.

I am also under obligation to Mr. W. G. Owen, Chief Engineer of the Great Western Railway, who kindly permitted me to choose examples from the practice existing on that line, as well as to Mr. Edmund Olander, of the same railway, for the readiness he shewed in supplying me with information concerning the structures I wished to set as examples.

Whilst preparing the chapter on Wind Pressures, I have been able, through the courtesy of Mr. James Forrest, the Secretary, to examine the valuable documents in the possession of the Institution of Civil Engineers.

Throughout the work the language used has been carefully studied with a view to simplicity and clearness—all technical terms are rigorously defined and systematically employed in the same sense. Thus, it has been found convenient to use the term *angular point* to describe a point situate at the intersection of two or more lines. Grammarians greatly addicted to form might be tempted to criticise the accuracy of this expression; but I have thought it better to sacrifice scholastic niceness to a clear and scientific terminology. In this instance, the term *angle* is not sufficiently precise.

In one or two places I owe inspiration to the works of M. Maurice Levy and M. Bresse; but for the most part the problems have been worked out independently of books.

Historically, the branch of the subject entitled Graphic Statics owes its growth and development to the labours of Mr. Taylor, a practical draughtsman in the office of the well-known contractor, Mr. J. B. Cochrane; to the late Professor Clerk Maxwell, who moulded the matter into more rigorous form; to Professor Fleeming Jenkin, who commenced the application of the subject to engineering structures; and to Bow, who in his work on *The Economics of Construction* has

given a vast number of roof examples accompanied by reciprocal diagrams. Bow has done much to extend the application of this new method in placing before those who have thoroughly mastered the subject a vast collection of roof examples, many of which must have presented great difficulty of solution; and again, by the introduction of the method of lettering explained in Chapter II. of the First Part of this work.

The letter-system is useful in the treatment of small structures embracing few members; but it becomes somewhat unwieldy of application to large structures containing one or two hundred independent parts.

The method of marking the corresponding lines of the structure and reciprocal figure by identical numbers enjoys three great advantages:—firstly, it enables the engineer to seize at a glance the stress-line belonging to any particular member of the roof or bridge he may wish to design; secondly, it obviates the necessity, when dealing with large and important structures, of affixing distinguishing marks to the limited number of letters in the English alphabet; and thirdly, as will be seen in the course of this work, it facilitates the construction of polar or funicular polygons.

As regards the development abroad, mention should be made of the labours of Culmann, Cremona, and Von Ott; and in a more special manner of those of Maurice Levy, who, on account of the clearness and lucidity of his style, has contributed, perhaps more than any other writer, to the spread of the science of Graphic Statics.

In conclusion, I believe I have a right to claim that here for the first time the twin subjects of Graphic and Analytic Statics have been brought together, compared, and exhaustively treated in the same work.

R. H. G.

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GRAPHIC AND ANALYTIC STATICS.

PART I.

GENERAL GRAPHICS.

CHAPTER I.

RECIPROCAL DIAGRAMS OF STRESS.

§ 1. *Geometric Definition of the terms Polygon of Forces, and Polar Polygon.*

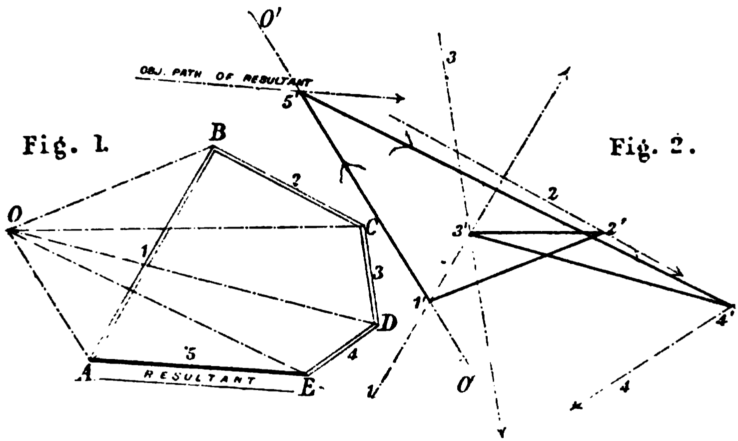
LET the lines, 1, 2, 3, 4, Fig. 2, represent the lines of action of a number of forces applied to the parts of any structure, and let the arrow-heads indicate the senses in which these forces are supposed to act.

If a second figure be constructed, having each of its lines parallel to one or other of these forces, and of a length proportionate to the intensity of the force;—this second figure will constitute what has been termed *the Polygon of Forces.*

For example, commencing at any point, A, Fig. 1, let us draw the line, $\overline{AB}$, parallel to force-line, 1, Fig. 2, and representing by its length and direction the intensity and sense of the chosen force. Similarly, from, B, the extremity of line, $\overline{AB}$, let the line, $\overline{BC}$, be drawn, parallel to the direction, and of a length proportionate to the intensity, of the force applied along the path indicated by the arrow-headed line, 2, Fig. 2. Continuing the same process, draw a line, $\overline{CD}$, parallel to, and

representative of the force, 3; and lastly draw the line, $\overline{D\,E}$, corresponding to the fourth force shewn in, Fig. 2.

The unclosed polygon thus formed and marked, $\overline{A\,B\,C\,D\,E}$, is called the polygon of the given forces; because each of its sides is a graphic representation of two special properties belonging to one of the applied forces; viz., its direction and intensity. But each applied force possesses a third property not given in this polygon. This third quality, which we may term the *actual path of application* of the force, is sufficiently defined by the arrow-headed lines in Fig. 2.



If, in order to complete the polygon of forces, $\overline{A\,B\,C\,D\,E}$, a line, $\overline{A\,E}$, be drawn joining its loose ends; it can easily be proved that this line will be a graphic delineation of the resultant sense and magnitude of the applied four forces. For, manifestly, the resultant of the first two forces, $\overline{A\,B}$ and $\overline{B\,C}$, could be represented, according to the well-known principle of the triangle of forces, by the line, $\overline{A\,C}$, connecting the *angular points*, A and C.

Again, replacing the first two forces by their graphic resultant, $\overline{A\,C}$, it follows by the same principle that the resultant of the forces, $\overline{A\,C}$, and, $\overline{C\,D}$, will be defined in direction and intensity by the line, $\overline{A\,D}$. Finally, the resultant of the forces, $\overline{A\,D}$ and $\overline{D\,E}$, will be represented by the line, $\overline{A\,E}$. But, by

implication, the line, $\overline{AD}$, is the graphic sum or resultant of the forces, $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$; therefore the line, $\overline{AE}$, will be the graphic sum or resultant of the forces, $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DE}$; or, in other words, the line, $\overline{AE}$, measured in the sense of from, A to E, is a graphic delineation of the resultant magnitude and direction of the applied four forces.

§ 2. *Definition of Polar Polygon.*

Let any point, O, Fig. 1, be arbitrarily chosen in the plane of forces, and from this point as origin let there be drawn five dotted lines, $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$, $\overline{OE}$, to the five *angular points* of the polygon of forces.

Next, returning to Fig. 2, draw any line, $O'O'$, across the force-line marked, 1, and parallel to the polar-line, $\overline{OA}$, of Fig. 1. Let the line, $O'O'$, intersect the force-line, 1, in a point, $1'$. From point, $1'$, so determined draw a second line parallel to the polar-line, $\overline{OB}$, cutting force-line, 2, in a point, $2'$. From, $2'$, draw a third line, $2'3'$, parallel to the polar-line, $\overline{OC}$, crossing force-line, 3, in a point, $3'$. Similarly, draw a line, $3'4'$, parallel to line, $\overline{OD}$; and lastly a line, $4'5'$, parallel to line, $\overline{OE}$, intersecting the line, $O'O'$, first drawn, in a point, $5'$.

The polygon, $1'\ 2'\ 3'\ 4'\ 5'\ 1'$, given by the preceding construction is called the *polar*, or *funicular polygon* of the given system of force-lines. Since the position chosen for the pole, O, is quite arbitrary and can be pitched anywhere in the plane, it follows that for the same system of force-lines there can be constructed relatively to different poles an indefinite number of polar polygons.

Further, it will be remembered that the line, $O'O'$, Fig. 2, was drawn parallel to the line, $\overline{OA}$, which joins the pole, O, to the origin, A, of the polygon of forces; but the position of the line, $O'O'$, was not further or more definitely restricted. In other terms this line can be moved parallel to itself into any other position in the plane. Consequently, since the directions of the lines composing the polar polygon are determined for any particular pole, O, by the directions of lines issuing

from that pole and passing to the *angular points* of the polygon of forces; it follows that by choosing the first line, $O' O'$, in various positions in the plane, there can be constructed a series of *similar polar polygons*, all related to the same pole, and having their sides respectively parallel to each other.

All polar polygons, whether drawn for the same or different poles, possess in common the singular property of indicating a point on the line of action of the resultant of the given system of forces. For instance, if a system of forces be applied to any structure in the objective paths, 1, 2, 3, and 4, Fig. 2;

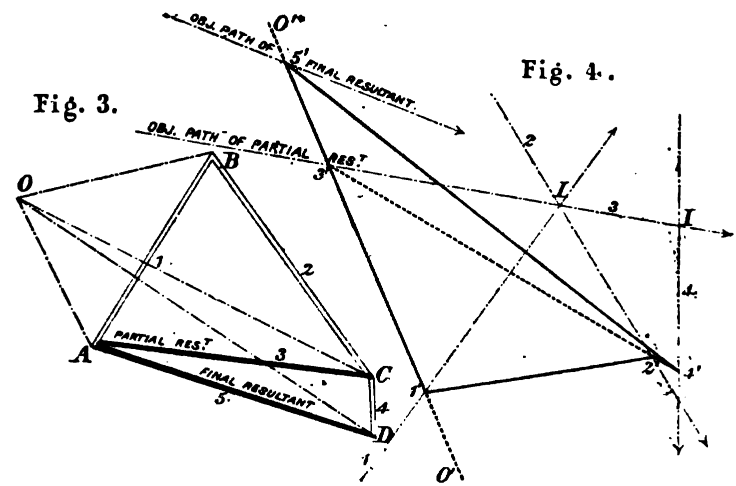


Fig. 3. Fig. 4.

their resultant will pass through a point, $5'$, determined by the intersection of the first and last lines, $O' O'$ and $4' 5'$, of the polar polygon. This statement will be demonstrated in the next article. In addition, since the polygon of forces, Fig. 1, gives the direction and magnitude of the resultant as represented by the line, $\overline{A E}$, of that figure; it may be inferred that the three properties of the resultant-force, expressing its sense, magnitude, and line of action, are furnished by means of the two figures, denominated the *polar polygon* and the *polygon of forces*. Hence, a line drawn through the point, $5'$, Fig. 2, parallel and equal to the line, $\overline{A E}$, Fig. 1, will represent not only the sense and magnitude of the resultant-force, but also the *actual path of its application*.

§ 3. *The Polar Polygon indicates a point on the Line of Action of the Resultant-Force.*

The demonstration of the theorem to be proved in this article is readily discovered by an examination of the most elementary form assumed by the polar polygon, in the case when the applied forces are reduced to two in number; and afterwards the proof can be extended to any system, and to any number of applied forces.

In, Fig. 4, let the lines, 1 and 2, with arrow-heads, delineate the objective paths of the given two forces. Fig. 3, shews the corresponding polygon of forces, $\overline{ABC}$, in which the lines, $\overline{AB}$ and $\overline{BC}$, represent in sense and magnitude the forces applied along the lines, 1 and 2, in Fig. 4. As before, the line, $\overline{AC}$, joining the loose extremities of the polygon, $\overline{ABC}$, will graphically define the direction and intensity of the resultant force.

With a view to the construction of the polar polygon of these two forces, pitch a pole at any point, O, Fig. 3, and from it draw lines to the three points, A, B, and C.

By the process already explained for Figs. 1 and 2, determine the lines of the polar polygon by first drawing any line, $O' O'$, Fig. 4, parallel to polar-line, $\overline{OA}$, Fig. 3. Let this line intersect the force-line, 1, in a definite point, $1'$. From point, $1'$, so found draw the line, $1' 2'$, parallel to the polar-line, $\overline{OB}$, and intersecting the force-line, 2, in a point, $2'$. Finally, through the point, $2'$, draw a line, $2' 3'$, cutting the line, $O' O'$, first drawn in a point, $3'$. Now, it is our object to prove that the resultant of the two applied forces must pass through the point, $3'$, found by the aid of the polar polygon.

In the first place, it is abundantly clear that the resultant of the two forces must pass through their common point of intersection, I, Fig. 4, and, further, it is manifest that the direction of this resultant must be parallel to the line, $\overline{AC}$, Fig. 3, which connects the loose ends of the polygon of forces, $\overline{ABC}$. Consequently it is only necessary for us to shew that a straight line, $3' I$, joining the points, I, and $3'$, is parallel to the line, $\overline{AC}$, Fig. 3. This parallelism being proved, it

follows by direct inference that the point, 3′, is a point located on the line of action of the resultant force.

As a premise to the proof, it will be generally admitted that five lines are sufficient to determine the form of a figure of six sides, made up of the six lines joining any four fixed points in the same plane. For example, the six-sided figure formed by the lines, joining the four fixed points, 3′, 1′, 2′, and I, Fig. 4, is fully defined before the sixth or last line, connecting the points, 3′ and I, has been drawn in ; for this line, 3′ I, which completes the figure, must necessarily traverse the apices, 3′ and I, of the component triangles, 3′ 1′ 2′, and 1′ 2′ I. Hence, if the five sides of Fig. 4, already drawn are respectively parallel to the five sides of Fig. 3 ; the sixth side, 3′ I, must be parallel to the sixth side, $\overline{AC}$, of the corresponding figure ; that is to say, the parallelism of the first five sides determines the parallelism of the sixth side ; just as the position of the first five sides has been shewn to involve that of the sixth side. But five sides of Fig. 4 *have* been constructed parallel to five sides of Fig. 3 ; that is—

1°. Line, 1′ 3′, (O' O'), has been made parallel to line, $\overline{OA}$;
2°. „ 1′ 2′ „ „ „ „ $\overline{OB}$;
3°. „ 2′ 3′ „ „ „ „ $\overline{OC}$;
4°. Line, 1′ I (force-line 1) is parallel to line $\overline{AB}$;
5°. Line, 2′ I (force-line 2) „ „ $\overline{BC}$;

Hence, line, 3′ I, must be parallel to line, $\overline{AC}$. *Q. E. D.*

Having, therefore, proved that the point, 3′, found by means of the polar polygon, is in reality a point situate on the line of action of the resultant force, $\overline{AC}$, it remains to extend the proof to any given system of forces.

For this purpose, let a third applied force be introduced, and supposed to act along the path, 4, Fig. 4 ; the sense of the force being as usual indicated by the arrow-head.

In the first instance, instead of considering the two forces, 1 and 2, separately, it will be assumed that they are graphically combined, and represented by their resultant, $\overline{AC}$, Fig. 3, which has been previously drawn in the polygon of forces.

Taking, then, the line, $\overline{AC}$, to represent the first two forces, it will now be necessary to draw another line, $\overline{CD}$, parallel to the third force recently added to the system. The line, $\overline{CD}$, must be drawn so as to indicate the direction and magnitude of the additional force, 4.

The figure, $\overline{ACD}$, so constituted, will form the polygon of forces in connexion with the forces, 3 and 4.

Observing the usual order, let us construct the polar polygon of the force-lines, 3 and 4. To this end pitch a pole anywhere in the plane, in which the forces are supposed to act. It will contribute to the clearness of the proof and construction, if the same pole be retained for the forces, 3 and 4, as was used in the case of the forces, 1 and 2. Consequently, from the pole, O, Fig. 3, draw lines to the three corners of the polygon of forces, $\overline{ACD}$. Two of these polar lines, $\overline{OA}$ and $\overline{OC}$, have been previously drawn in relation to the polygon, $\overline{ABC}$, and the forces, 1 and 2; and they are in fact polar-lines common to the two polygons, $\overline{ABC}$ and $\overline{ACD}$. It is, therefore, only necessary to draw the third polar line, $\overline{OD}$.

Next, construct the polar polygon of the forces, 3 and 4, relatively to the pole, O. For this purpose draw any line across the force-line, 3, parallel to line, $\overline{OA}$, Fig. 3. For the sake of simplicity in the demonstration this first line of the polar polygon has been made identical with the first line, $O'O'$, of the polar polygon, $1'\,2'\,3'$, previously constructed for the forces, 1 and 2. It is quite permissible to establish this identity; for, in the first place, line, $O'O'$, is by definition placed *anywhere* in the plane of applied forces; secondly, it is drawn parallel to line, $\overline{OA}$, which is a polar line *common* to the two polygons of forces, $\overline{ABC}$ and $\overline{ACD}$.

Let, then, the line, $O'O'$, drawn across force-line, 3, meet it in a point, $3'$. From point, $3'$, draw a line, $3'4'$, parallel to polar line, $\overline{OC}$, crossing line, 4, in a point, $4'$; and lastly through, $4'$, draw a line, $4'\,5'$, parallel to polar line, $\overline{OD}$, meeting the line, $O'O'$, first drawn, in the point, $5'$.

It will have been observed that the line, $3'4'$, coincides in part with the line, $3'2'$, belonging to the polar polygon of the forces, 1 and 2. The reason of this coincidence is that both

lines pass through the same point, 3′, and are drawn parallel to the common polar line, $\overline{OC}$.

It can be shewn by the method already explained with reference to the forces, 1 and 2, and which holds in all cases where the applied forces are only two in number, that the point, 5′, found above, will be a point situate on the line of action of the resultant of the forces, 3 and 4. Now, force, 3, is a partial resultant, representing the components, 1 and 2; whence, it follows that point, 5′, is located on the resultant objective path of forces, 1, 2, and 4.

To complete the proof, it remains to shew that the point, 5′, can be equally well determined by the polar polygon of the *three* forces, 1, 2, and 4, as it was by means of the polar polygon of the equivalent *two* forces, 3 and 4.

The construction of this third polar polygon is already contained in the figure; and, indeed, it is made up of two parts, one of which results from the polar polygon of the forces, 1 and 2, the other being comprised in the polygon last drawn in connexion with the forces, 3 and 4.

It will, perhaps, be advisable to repeat the process of construction of this compound polygon, as the details of the independent construction will at once define what parts are respectively derived from the two polygons previously drawn.

Retaining the same pole, *O*, Fig. 3, draw the polar lines, $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$. Across the force-line, 1, Fig. 4, draw any line, *O′ O′*, parallel to polar-line, $\overline{OA}$, cutting the line, 1, in a point, 1′. From point, 1′, thus determined, draw a line, 1′ 2′, parallel to line, $\overline{OB}$, intersecting force-line, 2, in a point, 2′. Through, 2′, draw a line, parallel to polar-line, $\overline{OC}$, and crossing force-line, 4, in a point, 4′. Finally, from, 4′, draw a line, 4′ 5′, parallel to, $\overline{OD}$, intersecting the line, *O′ O′*, first drawn, in a point, 5′.

The above construction will serve to shew that the dotted line, 3′ 2′, divides the polar polygon, 1′ 2′ 4′ 5′ 1′, into two parts; 3′ 1′ 2′ and 2′ 4′ 5′ 3′, the first part, 3′ 1′ 2′ being derived from the polar polygon, 3′ 1′ 2′, previously constructed for the forces, 1 and 2; and the second part, 2′ 4′ 5′ 3′, being borrowed

from the polar polygon, $3' 4' 5'$, of the forces, 3 and 4. A remark already made may be here profitably repeated, to the effect that the line $3' 4'$ is of necessity continuous, this continuity being insured by the fact that the two lines, $3' 2'$ and $2' 4'$, are drawn parallel to the same polar line, $\overline{OC}$.

The preceding demonstration virtually proves that the point, $5'$, can be reached by a separate consideration of the forces, 1, 2, and 4, as well, and as easily as by only taking into account the forces, 3 and 4. Evidence of this fact was necessary to complete the proof, and it has been shewn that the point, $5'$, can be come at by following two different paths. By one route the course open is indicated by the zigzag line, $0' 3' 4' 5'$, which results from the construction of the polar polygon corresponding to the forces, 3 and 4. By the alternative route, the course to be pursued is marked, $0' 1' 2' 4' 5'$, and defined by the polar polygon of the forces, 1, 2, and 4. In both cases the point, $5'$, is the objective goal or terminus, to which the two paths finally lead.

In a similar way a fourth force might be added to the system, and combined with the resultant, $\overline{AD}$, of the forces, 1, 2, and 4. The method explained for two applied forces would again become applicable, and in this manner the demonstration could be extended to any number of forces. Hence, it may be generally concluded, *that the polar polygon indicates a point on the line of action of the resultant of any system of applied forces.*

§ 4. *Geometric Definition of Reciprocal Figures.*

On a comparison of the two figures, marked, 3 and 4, a certain reciprocity will be found to exist between the lines composing them. For instance, parallel to the three lines $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$, branching from the pole, O, Fig. 3, there are in Fig. 4, three corresponding lines forming a triangle, $3' 2' 1'$; and, as in geometrical phraseology, two triangles are said to be *similar*, when their lines are respectively parallel or perpendicular to each other; so, in the language of graphic statics, two figures are said to be *reciprocal*, when the lines composing

them fulfil the two following conditions; 1° that their lines are respectively parallel or perpendicular to each other; 2° that lines radiating from a point in one figure are parallel or perpendicular to corresponding lines forming in the other a closed polygon. In future, figures fulfilling simultaneously the preceding two conditions will be termed *reciprocal figures*.

It will be seen, on a brief examination, that the figures 1 and 2; 3 and 4, are in the fullest sense reciprocal; for, in the first place, the lines composing them have been drawn parallel to each other; secondly, corresponding to any point or pole, B, Fig. 3, out of which branch three lines, $\overline{OB}$, $\overline{AB}$, and $\overline{BC}$, there exists in Fig. 4, a closed figure, $1'\,2'\,I$, having its sides respectively parallel to the lines, which radiate from the nucleus, B. Again, corresponding to the four lines flowing from the pole, O, Fig. 3, a closed figure will be found in Fig. 4, complying with the requirements specified in the given definition. This closed figure will be at once recognised as the polar polygon relative to the pole, O, of the forces, 1, 2, and 4; and marked, $1'\,2'\,4'\,5'\,1'$.

In practice, the problem to be solved generally consists the construction of the closed polygons, reciprocal of the poles or nuclei of a class of figures, resembling, Fig. 4. As an example, take the point, $1'$, Fig. 4, which is a nucleus or pole, and must have for its reciprocal in Fig. 3, a closed figure. This closed figure must have its lines respectively parallel to lines, $1'\,3'$, $1'\,I$, and $1'\,2'$, which radiate from point, $1'$. It will be found that the triangle, $\overline{AOB}$, Fig. 3, fulfils all these conditions. Similarly, the triangle, $\overline{OBC}$, is the reciprocal figure, corresponding to nucleus, $2'$, Fig. 4.

§ 5. *Criteria.*

The criteria, by the aid of which it may be decided whether or not any given structure admits of graphic treatment, may be classed under three heads.

1°. The base figure, on which the science of graphic statics is built up, is the triangle. Now, the sides and *angular points*

of all triangles bear to each other a certain definite relation, which can be expressed in the following form :

$$S = 2p - 3$$

in which formula,

$S =$ the number of sides of the triangle
$p =$ the number of its angular points.

In all figures, however complex, which are subject to graphic treatment, the above relation between the number of sides and nuclei, forming part of the essential framework of the structure, will be found to exist. Many designs, however, in which this relation is non-existent, can be brought under the laws of graphic statics by rejecting certain bars or other elements, which do not contribute to the *essential*, but only to the *accessory* form of the structure. A certain mathematical instinct, or rather a species of intuition acquired only by habit, is necessary to distinguish these cases.

2°. Although the triangle constitutes the base of the graphic system, strange to say it is not a figure, which *per se* admits of a reciprocal. This will be at once apparent from the fact that from each apex of a triangle branch only two lines, and since it is impossible that any two lines parallel to these can of themselves form a closed figure ; the conclusion is forced upon us that, for any figure to become directly subject to graphic treatment, *there must radiate from each of its nuclei at least three lines.*

3°. The third condition may be stated to imply the necessity, that each line composing the given figure should traverse two nuclei ; otherwise the loose end, which did not belong to any pole or nucleus, would *ipso facto* become indeterminate in position ; or, in more general terms, the reciprocal line would be indeterminate in length, and therefore could not occupy a definite place in the reciprocal figure.

This article may be concluded with the remark that in Figs. 3 and 4, with the exception of the auxiliary line, 3′ 2′, the reciprocals of the full and dotted lines of Fig. 3 are shewn in Fig. 4 by dotted and full lines, and *vice versâ ;* the reciprocal of each full line of Fig. 4 is given as a dotted line in Fig. 3.

§ 6. *Terminology.*

In this article a certain convenient terminology will be explained, which has been used to distinguish the lines of reciprocal figures. These conventional symbols will be best understood by taking an example.

The line, $\overline{OB}$, Fig. 1, is the reciprocal of the line, $1' 2'$, Fig. 2, and the point, B, forms what may be conveniently called the *junction* of lines, $\overline{AB}$ and $\overline{BC}$; or, taking the same two lines as marked in figures instead of letters, we may say that point, B, is the junction of lines, 1 and 2, Fig. 1. The line, $\overline{OB}$, may consequently be designated by help of the symbol, $O_{1\cdot 2}$, which is thereby defined to mean a line, drawn from the pole, O, to the *junction* of the lines, 1 and 2. Similarly, line, $\overline{OC}$, can be expressed as polar-line, $O_{2\cdot 3}$; line, $\overline{OD}$, as $O_{3\cdot 4}$; line, $\overline{OE}$, as $O_{4\cdot 5}$, and so on.

According to the above definition the line, $O' O'$, which has been drawn parallel to, $\overline{OA}$, would be called the reciprocal of, $O_{1\cdot 5}$; and the line $1' 2'$, which has been drawn parallel to, $\overline{OB}$, would be termed the reciprocal of, $O_{1\cdot 2}$. Similarly, the line, $2' 3'$, is the reciprocal of $O_{2\cdot 3}$; the line, $4' 5'$, the reciprocal of, $O_{4\cdot 5}$; and lastly, line, $5' 1'$, is the reciprocal of, O_{51} or O_{15}, both of which symbols imply the same line, $\overline{OA}$. We shall in future make use of these conventions on account of their simplicity, and the identity which exists in the numbers applied to reciprocal lines; as for example in the similarity of the numbers involved in the description of the lines, $O_{1\cdot 2}$ and, $1' 2'$, which are evidently reciprocal.

§ 7. *Examples Worked Out.*

1. A WARREN GIRDER.—In working out examples by the graphic method certain rules have to be observed which will be here explained. These rules will be best learnt and understood by the help of an example.

Let Fig. 5 represent in skeleton outline an ordinary Warren girder, subjected to certain definite vertical loads acting along

the objective paths indicated by the force-lines, 1—8, which loads are supposed to be borne independently by the lower joints of the girder.

The next part of the process, after delineating the skeleton outline of the structure, and defining by lines the objective paths of the applied forces, is to construct the polygon of these forces.

In the present case, the forces, being all vertical, will all take the same direction, and consequently the polygon of forces will be reduced to a straight line, 1—8, Fig. 6, of a length proportionate to the sum-total of the eight separate loads. On this line, commencing at the top, as shewn in the figure, set off all the loads in their due order; so that line, 1, Fig. 6, is proportionate to the part-load applied at joint, 1, Fig. 5; line, 2, proportionate to the load at joint, 2; and so on down to the last division of the load line, marked, 8, which expresses by its length the proportion of the load applied at joint, 8.

We next proceed to the graphic determination of the reactions, due to the supports at 9 and 10. From any point, O, Fig. 6, as a pole, draw a series of diverging or polar lines to meet the vertical line of loads at points corresponding to the eight divisions of which it consists. These lines are shewn in the figure. Construct, now, the reciprocal, closed figure, relatively to the pole O, of the given system of force-lines; that is to say, draw across force-line, 1, Fig. 5, a line, 1′ 10′, parallel to, $O_{1.10}$, Fig. 6. From the point, 1′, so determined, draw a line, 1′ 2′, parallel to line, $O_{1.2}$, cutting the force-line, 2, in a point, 2′. Continuing the same process, draw lines, 2′ 3′; 3′ 4′; 4′ 5′; 5′ 6′; 6′ 7′; 7′ 8′; and 8′ 9′, respectively parallel to lines, O_{23}; O_{34}; O_{45}; O_{56}; O_{67}; O_{78}; and O_{89}. It will be seen that the line, 1′ 10′, first drawn, intersects the line of reaction, 10, Fig. 5, in a definite point, 10′; and in like manner the line, 8′ 9′, last drawn, meets the line of reaction, 9, at a point, marked, 9′. Join these two points by a straight line, 9′ 10′; and in Fig. 6, draw a reciprocal line, $\overline{O\,z}$, parallel to line, 9′ 10′, dividing the vertical line of loads into two parts, 9 and 10. The upper part, 10, will represent the amount of reaction

taking place at support, 10; and the lower part, 9, that occurring at support, 9.

The addition of the two reactions, just made, completes the vertical line of loads, and transforms it into a *double line*, made up on one side of the separate loads and on the other of the two reactions. It will be, further, observed that this double line is in reality self-closing; for, starting from its upper end and reading downwards, there occur in succession the various applied loads, 1 to 8;—then returning and reading in an upward sense, we are led, by help of the two reactions, back to the same starting-point at the upper end of the line. This is only the natural result of the forces applied to the girder being in equilibrium. Wherever equilibrium exists, the polygon of forces or the line of forces, as the case may be, must necessarily close. The polygon just drawn and traversing the force-lines of Fig. 5 forms the polar polygon, relatively to the pole, O, of the given system of applied forces, and it has the singular power, already proved (§ 3), Part I., Ch. I., of indicating a point on the line of action of the resultant-force of the system. In the present example, the forces being in equilibrium, there can be no actual resultant-force. Nevertheless, the *path* in which the resultant would act, in case the loads were unresisted at the supports, implicitly exists and can, therefore, be found as usual by aid of the polar polygon. A point situate on this path will be determined by the production of the first and last lines of the polar polygon, $1'\,10'$ and $8'\,9'$, which intersect at, I. This point will form an objective locus in the plane of the applied forces, through which the resultant force would act in the absence of any reaction. Moreover, since the resultant of vertical forces acts in a vertical line, the line, II', will constitute the line of action of the ideal resultant of the given system.

If the vertical line of loads be divided at, z'; so that, $\frac{xz'}{z'y} = \frac{I'B}{I'A}$; the points, z' and z, will be found to coincide, which proves that the line, Oz, drawn parallel to, $9'\,10'$, correctly divides the line of loads in the ratio of the reactions. The proof of this fact can be presented in another form. It

will have been observed that the lines, which branch from the pole, *O*, divide the line of loads into parts, corresponding to the separate values of the applied forces ; insomuch that any diverging line, O_{23}, is drawn to the *junction* of the loads, 2 and 3, on the line of forces. Moreover, the reciprocal of this polar-line falls between the force-lines, 2 and 3, Fig. 5, and is marked, 2′3′. By analogy, therefore, the reciprocal of the line, 9′10′, Fig. 5, which crosses the force-lines, 9 and 10, will form in Fig. 6, a polar line, drawn to meet the line of loads at the *junction*, *z*, of the reactions (9 and 10).

Examining the nucleus, 10, 11, 12, Fig. 5, it will be seen that this nucleus is a point under the influence of a force, 10, distributed in some unknown way along the bars, 11 and 12. Earlier in this work, a demonstration was given shewing that closed figures, reciprocal of nuclei, have, in common with the triangle of forces, the singular power of expressing the amount of the force or stress applied along the bars or lines branching from the point considered. Consequently in this case, to find the stresses induced in bars, 11 and 12, the reciprocal of the nucleus, 10, 11, 12, must be constructed.

By a former definition, this reciprocal figure, of the nucleus, 10, 11, 12, must fulfil two conditions ; 1°, the lines composing it must be respectively parallel to the lines, 10, 11, 12, issuing from the point or pole in question.

2°. It must form a closed polygon, which in this instance is reduced to a triangle (Part I. Ch. I. § 4).

One line of this reciprocal triangle already exists in Fig. 6 ; insomuch that line, 10, Fig. 6, measures the amount of reaction along the force-line, 10, Fig. 5. To complete the triangle required, two lines must be drawn from the ends of the known line, 10, and parallel respectively to the lines, 11 and 12, Fig. 5. The question here arises :—From which end of, 10, must these lines be respectively drawn ? This ambiguity is easily removed ; for it will be seen that in, Fig. 5, the lines, 1, 10, 11, and the line, 10′1′, of the polar polygon form a closed quadrilateral figure, and, therefore, their four reciprocals in, Fig. 6, will constitute a pole. Wherefore, from the junction of 10 and 1 on the line of forces, draw a line parallel

to bar, 11, Fig. 5, and from the other end of, 10, draw a line, 12, parallel to bar, 12. The triangle, 10, 11, 12, thus constructed is the reciprocal figure of the nucleus, 10, 11, 12, and its lines are a graphic delineation of the stresses acting along the bars meeting at that point.

There still remains one indeterminate element;—it is not directly known whether the stresses, as shewn by the reciprocal figure, are positive or negative; that is, whether the bars subject to stress are in tension or compression. To decide this point, it will be convenient to suppose the forces applied at the joint, 10, 11, 12, in the way shewn separately in Fig. 7; the *sense* of each force being derived by first going round the triangle of stresses, 10, 11, 12, Fig. 6, beginning with the known direction of reaction, 10. These forces must then be drawn with arrowheads indicative of their sense, as shewn in Fig. 7, and all supposed to be making for, or not yet arrived at the nucleus, *N*, or *A*. This being done, those forces will be compressive which fall upon and along the bars to which they are applied, and those which lie outside of their bars will be tensional stresses. According to this rule, the stress, 12, will be compressive, and, 11, tensional; because the arrow indicating the latter force is situate on the prolongation of bar, 11, and not on the bar itself.

Proceeding in order to the next nucleus, let us construct its reciprocal figure, which will, of course, be a closed quadrilateral. We know already by previous construction two out of the four lines composing it; viz., line, 1, Fig. 6, given on the line of forces, and line, 11, which formed part of the reciprocal just drawn. It only remains to construct the lines reciprocal of the bars, 13 and 14; in such a way that they may make with the known lines, 1 and 11, a closed polygon. Now, in Fig. 5, the bar, 13, forms with bars, 11 and 12, a closed figure; therefore, in the reciprocal, the same three lines will constitute a nucleus. Consequently, from the junction of, 12 and 11, in Fig. 6, draw a line, 13, parallel to the bar, 13, of Fig. 5. Similarly, and for the same reason, from the junction of, 1 and 2, on the line of forces, draw a line, 14, parallel to bar, 14, Fig. 5. The lines, 13 and 14, so

drawn, will intersect and form a closed quadrilateral figure, 1, 11, 13, 14. By the rule for tensions and compressions, 13, is the only compressed bar at this nucleus.

Proceeding next to the nucleus, (12, 13, 15, 16) it will be seen that we are already in possession of the lines, reciprocal of, 12 and 13 ;—and since line, 15, forms a closed figure with lines, 13 and 14, in Fig. 5, it will, in Fig. 6, constitute a nucleus with the same lines. Consequently, from the junction of lines, 13 and 14, in Fig. 6, draw a line, 15, parallel to bar, 15. Similarly; since the reciprocal of bar, 16, must necessarily form a closed figure with the reciprocals, 12, 13, 15, already drawn; from the end of line, 12, opposed to its intersection with, 13, draw a parallel line to bar, 16. In the reciprocal figure, (12, 13, 15, 16) just constructed, the component lines, 12 and 16, coincide in part; and on this account it must be particularly observed that the line, 16, extends from junction, (15, 16) to the end of line, 12, opposed to that at which it is met by line, 13, or, in other words, from the vertical line of loads.

A second remark is, moreover, needed concerning what lines are compressive and tensional at this joint. The *sense* of the stress, 12, as determined by the reciprocal of the joint, (10, 11, 12), is diametrically opposite to the sense defined for it by the reciprocal of the joint, (12, 13, 15, 16). This follows from the fact that the action and reaction along any bar are forces equal in magnitude but opposite in sense. Consequently the closed figure (12, 13, 15, 16), Figs. 6 and 8, must be read, beginning at the extremity of line, 12, opposed to its intersection with 13, and following the sense indicated by the arrows (Fig. 8).

By the ordinary rule, 15, is the only bar in tension at this joint.

The reciprocal polygons of the remaining joints are constructed according to the same principles, and the whole figure can be easily completed by anyone who has followed and understood the preceding construction.

To distinguish the bars under compression from those under tension, the former have been drawn in shaded lines.

Discussion of the Stresses.—Referring to the figures, 5 and 6, we learn that compression reigns all along the upper horizontal line of bars and goes on increasing towards the middle of the girder; or rather towards that point in which the upper horizontal line is intersected by the objective path, *I I′*, of the virtual resultant force. This compressive stress reaches a maximum along the bar, 24.

On the other hand, tension obtains along the lower line of the girder, the maximum occurring in bar, 22.

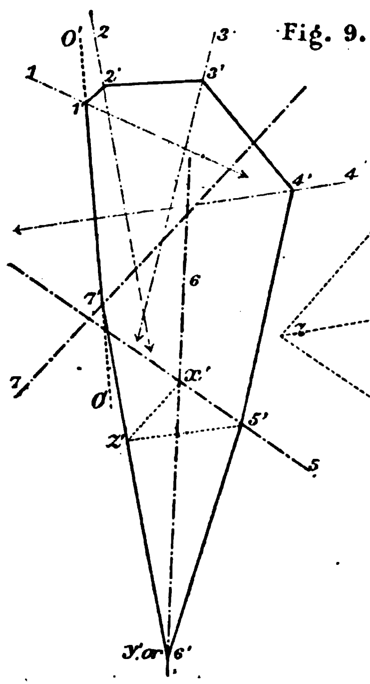


Fig. 9.

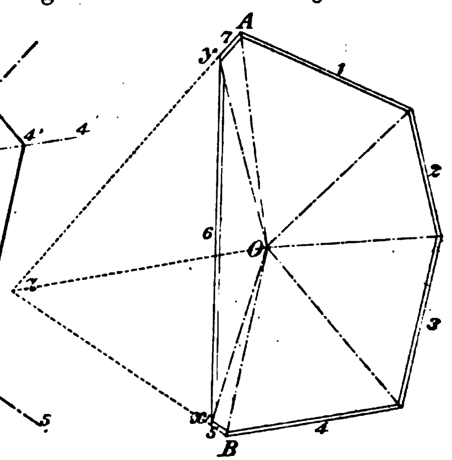
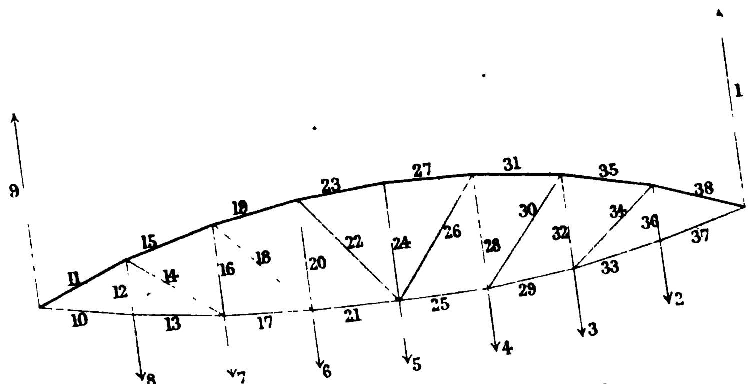


Fig. 10.

The inclined bars form ties and struts, being alternately under tension and compression, with the exception of the bars, 23 and 25, which are both under tension. The reason of this anomaly is explained by the fact that the resultant load passes through the space enclosed between these two bars.

As a general rule, the inclined stresses are greatest in those bars which are situate at the ends of the girder, decreasing in

Fig. 11.



1
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11
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19
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26
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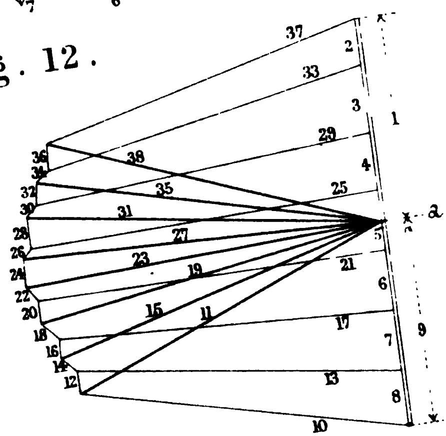
8

Fig. 12.



37
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a
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12

intensity as the bars become less and less distant from the locus of the virtual resultant load. The maximum inclined tension takes place in bar, 11; and the maximum inclined compression in the bar, 13. This is due to the fact, that these bars come under the influence of the greater reaction at, 10.

The stresses, both horizontal and inclined, are greater on that part of the girder, which forms the lesser of the two divisions separated by the line of resultant load. If this line bisected the length of the girder, there would be on each side of it a symmetrical distribution of the stresses.

Another remark seems called for in reference to this case. At the outset, the reactions at the supports, 9 and 10, were unknown, and had to be determined by means of the polar polygon. The question, therefore, naturally arises:—How many unknown forces can be found by the help of Graphic Statics?

With a view to the solution of this problem, it will be remembered that in, Figs. 1 and 2, four forces were given in magnitude and direction; and that by the aid of the polar polygon a point, 5′, on the objective path of the resultant, was discovered. But the magnitude and direction of this resultant was found simply by the construction of the polygon of forces. In that instance the function of the polar polygon was exercised only to define a point in the ideal structure through which the resultant force acted.

We proceed to show that the functions of polar polygons can be further extended to determine *three* unknown forces; provided the system, to which they belong, be in equilibrium, and the objective paths be given in which the forces act.

Let the lines, 1, 2, 3, 4, Fig. 9, represent the objective paths of four known forces, and let, 5, 6, 7, indicate the lines of application of three unknown forces, or resistances.

Construct the polygon of forces, 1, 2, 3, 4, representing by its lines the graphic values of the four given forces.

In this polygon the force, 5, must follow immediately after the force, 4; that is to say, one extremity of its graphic equivalent will enter the junction, *B*, Fig. 10.

Again, the system being in equilibrium, the force, 7, may be

looked upon as the resultant of the other six forces; consequently, one end of its graphic equivalent will enter the origin, *A*, Fig. 10.

Wherefore, in Fig. 10, draw two lines, *Bz* and *Az*, the first parallel to the line of action of the force, 5; the second parallel to that of the force, 7. These two lines will intersect in a point, *z*; and the graphic equivalents of the forces, 5 and 7, will form respectively parts of these two lines.

It remains to find the position of the graphic equivalent of the force, 6, which, by intersecting the lines, *Az* and *Bz*, in two points, *y* and *x*, will render the graphic representations of the three unknown forces perfectly definite.

For this purpose, pitch any pole, *O*, in Fig. 10, and draw the polar lines, $\overline{OA}$; O_{12}; O_{23}; O_{34}; $\overline{OB}$; and Oz.

Next, construct the polar polygon, relatively to this pole, by drawing across the force-line, 1, Fig. 9, any line, $O'\ O'$, parallel to the polar line, $\overline{OA}$. Let this line intersect the force-lines, 1 and 7, in points marked, 1′ and 7′. From point, 1′, draw a line, 1′ 2′, parallel to polar line, $O_{1\cdot 2}$;—from point, 2′, a line, 2′ 3′, parallel to polar line, O_{23};—a line 3′ 4′ parallel to O_{34};—a line 4′ 5′ parallel to polar line, $O_{4\cdot 5}$, or $\overline{OB}$; and finally the indefinite line, 5′ *z*′, parallel to, Oz.

Now, let the lines of action of forces, 5 and 6, meet at a point, *x*′, Fig. 9; and let us suppose the path of force, 7, shifted parallel to itself, so as to pass through the point, *x*′, in the direction, *x*′ *z*′, and meet the line, 5′ *z*′ in *z*′. The point, *x*′, becomes the nucleus of the three force-lines, 5, 6, 7, the last, (7), being represented by a line parallel to its true path. Join the points, *z*′; 7′, by a straight line, and produce it to meet the force-line, 6, in a point, *y*′, or 6′.

Proceeding in due order, construct the reciprocal of nucleus, 7′, which will form a triangle having its sides parallel to the lines radiating from this point. It will be seen that two of the sides of this triangle; viz., lines, $\overline{OA}$ and *Az*, already exist in, Fig. 10, parallel respectively to the lines, 7′ 1′, and force-line, 7. Therefore, if from the pole, *O*, a line, *Oy*, be drawn parallel to line, 7′ *y*′; the triangle, *O Ay*, will be the reciprocal of the nucleus, 7′, Fig. 9.

Fig. 13.

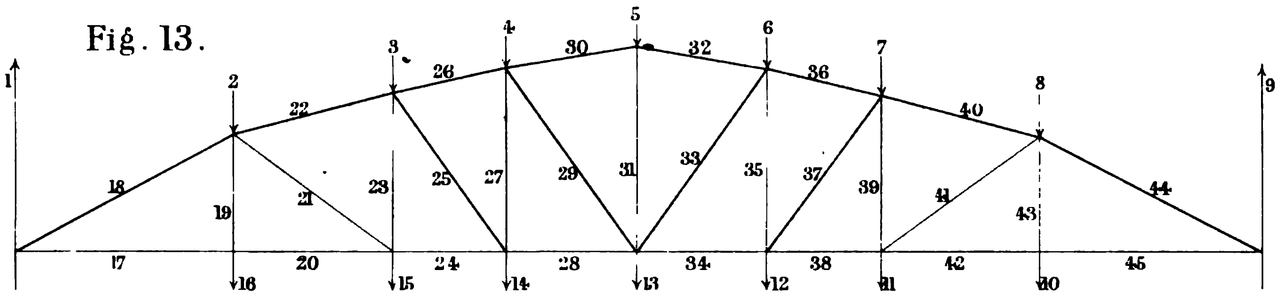
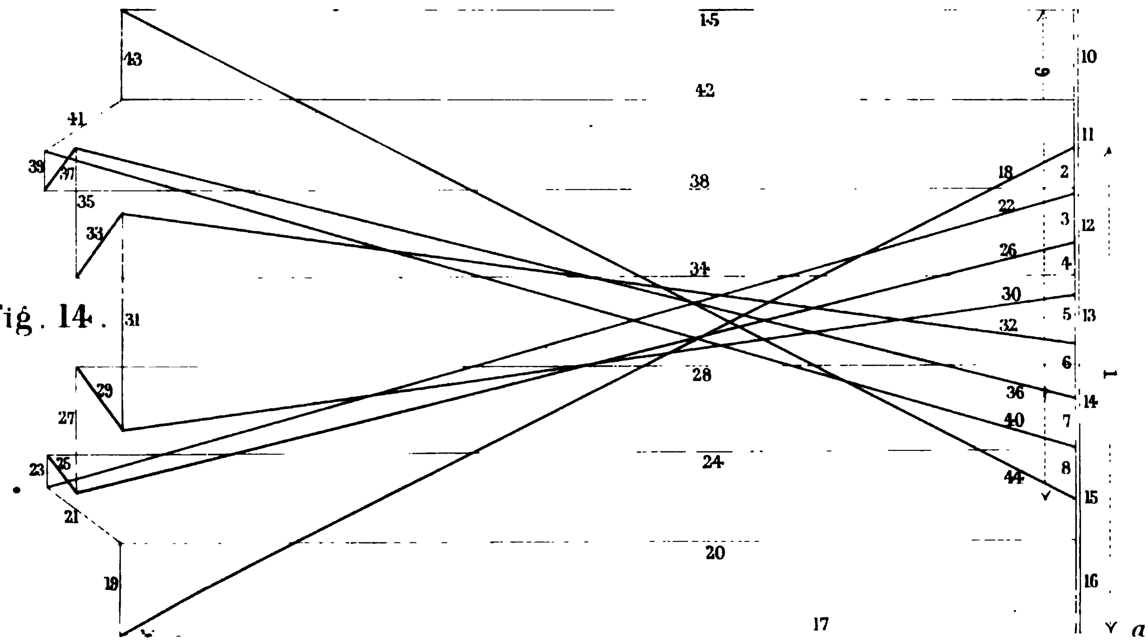


Fig. 14.



Proceeding next to the nucleus, 5′, we know the reciprocals of all the lines issuing from that point, with the exception of, 5′ 6′; which line, forming, in Fig. 9, a nucleus with lines, 4′ 5′ and force-line, 5, will compose with their reciprocals in Fig. 10 a closed figure. Consequently, if from the pole, O, a line $O\,x$, be drawn parallel to, 5′ 6′, the completed triangle, $O\,B\,x$, will constitute the reciprocal of nucleus, 5′, Fig. 9.

Passing in succession to the nuclei, x', y', z', it will be found that the reciprocals of all the lines branching from them are known and represented in Fig. 10, with the exception of the reciprocal of the line, 6; and as it happens that the five lines, which in Fig. 10 join the four fixed points, O; x; y; and z, are by construction the reciprocals of the five lines connecting the four fixed points, 5′; x'; y'; and z', Fig. 9; it follows from a previous demonstration (Pt. I. Ch. I. § 3) that the sixth line, $x\ y$, Fig. 10, joining the apices, x and y, is the reciprocal of the sixth line, $x'\ y'$, Fig. 9, connecting the apices x' and y', (line 6).

By the above method the magnitudes and directions of three unknown forces can be definitely determined; but, it may be added, the need of this graphic process is seldom felt in ordinary practice. A noteworthy example of its application is given in Part II. § 12, where it is used as an instrument to find the general reciprocal figure of the special roof-frame considered.

2. A Double Bow-string Girder:—In Fig. 11 is given the skeleton outline of a bow-string girder, the reciprocal figure being constructed according to the same principles as for a Warren-girder. That is, from any point, a, Fig. 12, chosen as origin, set up a line, 1, equal to the reaction, 1;—from the upper end of this line set off in succession and in a downward direction the loads, (2, 3, 4, 5, 6, 7, 8), which are supposed to be suspended from the lower chord. From the end of line, 8, draw a vertical line equal to reaction, 9;—this line ought to terminate at, a, and close the line of loads.

Commencing at the nucleus (9, 10, 11);—from the junction, (8, 9), set off a line parallel to bar, 10, and from junction, (1, 9), a parallel to bar, 11. The reciprocals, 10, 11, so drawn.

will intersect and form, with reaction-line, 9, a triangle constituting the reciprocal figure of nucleus (9, 10, 11).

Proceeding next to nucleus, (8, 10, 12, 13), from junction, (10, 11), draw a reciprocal line, 12, parallel to bar, 12; and between, (7, 8), a line, 13, parallel to bar, 13;—these two lines, (12, 13), will intersect and form a closed quadrilateral figure (8, 10, 12, 13), reciprocal of the nucleus similarly described. The rest of the construction is simply a repetition of that already given. It will be remarked that the bars of the upper chord are all in compression; whereas the diagonal members, uprights, and bars of the lower chord are all in tension.

3.* A Simple Bow-string Girder:—In Fig. 13, is given the skeleton-outline of a second form of bow-string, somewhat similar in design to the last example; but with a straight, in lieu of a parabolic under chord, and loaded upon the upper as well as upon the lower members.

The line of loads can be set out by commencing at any point, a, Fig. 14, as origin, and drawing a vertical line equal to reaction, 1;—then marking off in succession and in descending order the loads, (2, 3, 4, . . . , 8), applied to the upper chord;—thirdly, from the end of line, 8, setting up the reaction 9;—and lastly, from the upper end of 9, drawing in consecutive order the loads, (10, 11, ... 16), suspended from the platform. The line of loads so composed must close at the origin, a.

Commencing at the nucleus, (1, 17, 18), construct the reciprocal figure by the same method as given in the last example; that is, from junction, (1, 2), draw a line, 18, parallel to bar, 18; and from (1, 16) a line, 17, parallel to bar, 17.

Proceed next to nucleus, (16, 17, 19, 20), and draw from junction, (17, 18), a line, 19, parallel to bar, 19; and from, (15, 16), a line, 20, parallel to bar, 20.

Thirdly, passing to nucleus, (2, 18, 19, 21, 22), draw from junction, (19, 20), a line, 21, parallel to bar, 21;—and from, (2, 3), a line, 22, parallel to bar, 22. The lines, 21, 22, so

* Lévy, La Statique Graphique, where this example is similarly treated.

drawn, will intersect, and complete the closed polygon, (2, 22, 21, 19, 18), reciprocal of the nucleus similarly described.

The rest of the construction is similar to that already given, with the exception that after constructing the reciprocal of nucleus, (4, 26, 27, 29, 30), the sequence, hitherto observed in passing from upper to lower nuclei, must be broken, and the reciprocal of, (5, 30, 31, 32), drawn in before that of, (13, 28, 29, 31, 33, 34).

4.* A SUSPENSION BRIDGE:—In Fig. 15, is given the skeleton-outline of a suspension bridge, the determination of the stresses in which presents no special difficulty.

Construct first the polygon of forces,

1, 2, 3, ... 14, 15, 16,

corresponding to the series of loads applied at the joints of the platform, and to the end reactions, 15, and 16.

Next, beginning at the nucleus, (16, 17, 18), draw in the general reciprocal figure by rules already fully detailed in former examples.

5.† A SWIVELLING CRANE:—In Fig. 17, we have reproduced a good example of the application of graphic statics to the treatment of stresses in a swivelling crane, which is movable about the vertical axis, 11, and supported by the tension rod, 12.

The distributed load acting at the various joints of the braced jib is represented by the vertical forces,

1, 2, 3, 4, . . . 8, 9,

set off on the line of forces, *a d.*

A point on the line of action of the resultant of these nine forces can be determined by constructing, as shewn in dotted lines, their polar polygon relatively to the pole, *O*. Let the line, *R R*, represent this line of action. It will be seen that the load, *R;* or the sum of the given nine forces, is finally

* This example is given in a paper by Prof. Fleeming Jenkin, Trans. R. S. E., Vol. XXV.

† Le Figure reciproche nella Statica grafica (Cremona), where a similar treatment is given.

transmitted to the bars, 10, 11, and 12, at the base of the frame. The force, R, must, therefore, be decomposed, so as to find the tensions and compressions acting on each of these three bars.

For this purpose produce a line through bar, 10, meeting the path of, R, in a point, C. Join, C and i.

On the polygon of forces, draw a line, $a\,b$, parallel to line, $C\,i$, and a line, $d\,b$, parallel to bar, 10. The force, $R = a\,d$, is in this way resolved into two forces, $d\,b$, or 10, and $a\,b$, a force acting along the line, $C\,i$. This latter force, $a\,b$, can now be decomposed into two components; one, 12, acting along bar, 12; the other, 11, acting along bar, 11.

The remainder of the construction presents no difficulty; for, proceeding to nucleus, (10, 9, 27, 26), we know already the reciprocals of, 9 and 10, which are given on the polygon of forces. Moreover, in the skeleton-outline, bar, 27, forms a close figure or triangle with bars, 10 and 11;—hence, from the junction of lines, (10, 11), draw a line, 27, parallel to bar, 27; and from junction, (8; 9) a parallel to bar, 26. These two lines will intersect and form, with bars, 9, 10, a closed polygon, reciprocal, of nucleus, (9, 10, 26, 27);—and similarly for the rest of the figure.

Fig. 15.

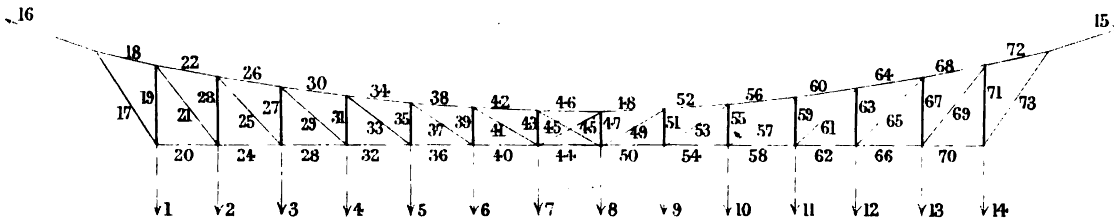
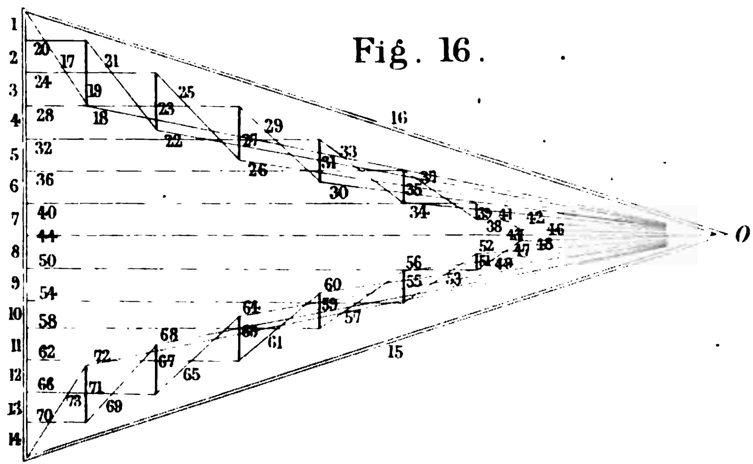


Fig. 16.



Tho.s Kell & Son Ltd

Fig. 17.

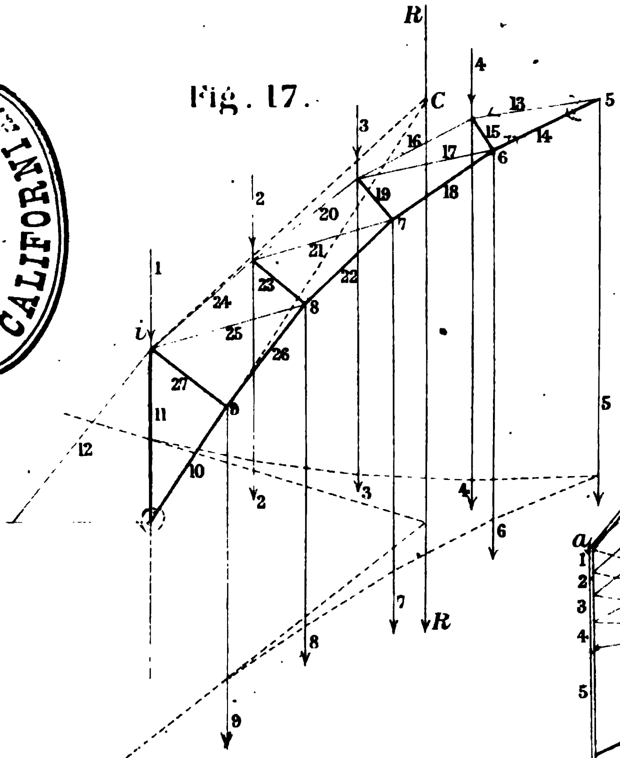
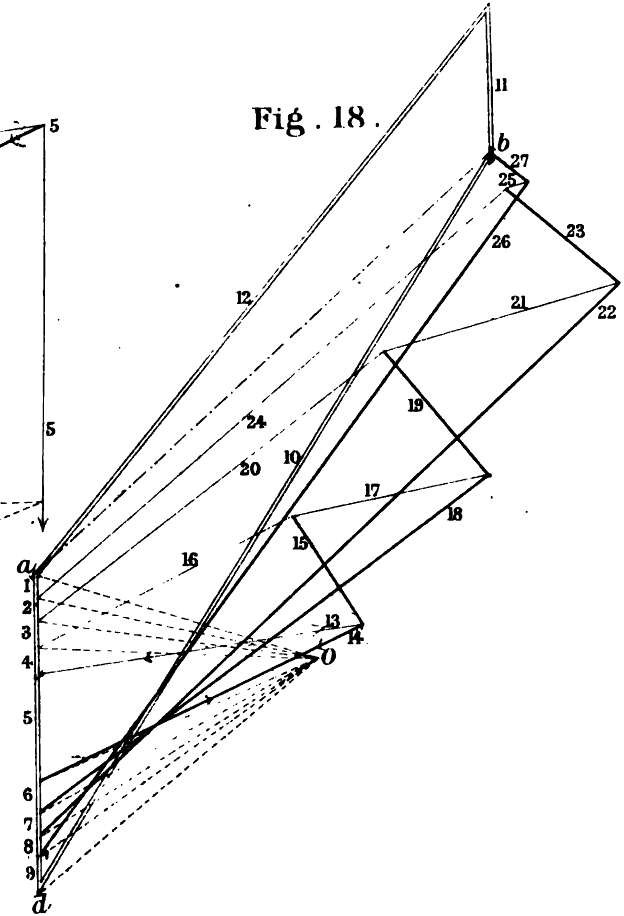


Fig. 18.



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CHAPTER II.

THE METHOD OF LETTERING THE INTERVENING SPACES.

THE system of lettering the spaces between the bars of the skeleton-outline of the given structure is considered by some writers on this subject preferable to the system of numbering the lines, adopted in this work.

According to the letter system, any individual member of the structure is defined by two letters which mark the spaces situate on both sides of it. Thus in* Fig. 19 the force acting at the ridge of the roof would be called, force, $c\,b$, and its reciprocal in Fig. 20, is the line contained between the same two letters; or, in other terms, the middle force, $c\,b$, on the vertical line of loads. Accordingly, therefore, the bars or members of the given structure are characterised by two letters, belonging to the two divisions of space, which lie on opposite sides of the line considered; and the reciprocals of these members are the lines situate between the points similarly marked on the diagram of stress.

The left-hand reaction in Fig. 19 would be called reaction, $d\,K$; because it lies between the spaces, d and K, and its reciprocal is the line-length, $d\,K$, Fig. 20.

It may be well to go through the construction of the reciprocal figure of this roof truss, on the letter system; so as to familiarise our readers with its use.

Commencing as usual with the line of loads or forces, take any point, d, Fig. 20, and draw an indefinite line, representing

* The figures used as illustrations in this chapter are taken, with slight modifications in the method of lettering, from the papers of Prof. Fleeming Jenkin (see Royal Trans. Edin.).

the general direction of the loads, which, in this case, is vertical. On this line set off in succession the forces, *d c*, *c b*, *b a*; returning to *d* by means of the equal reactions, *a K*, *K d*. The line of forces is now complete.

To draw the rest of the figure, begin at the left-hand nucleus, *d e K*;—which is thus described by the several divisions of space about this point,—and, observing that we have already drawn the reciprocal of the reaction, *d K*, complete the reciprocal triangle, *d e K*, Fig. 20, by drawing from point, *d*, a line parallel to bar, *d e*, and from, *K*, a second line *K e*, parallel to bar, *K e*. These two lines will intersect at, *e*, and form a triangle reciprocal of the nucleus, *d e K*.

It may be here remarked that this method possesses, from one particular point of view, a certain advantage over the method of numbering; insomuch that the mere description of the bar, *d e*, at once informs us that its reciprocal must be drawn from a point, *d*, already fixed in the diagram of stress. Similarly, the description of the bar, *K e*, tells us that its reciprocal must be drawn from point, *K*. Moreover, since both descriptions contain the common letter, *e*, which marks the space, intermediate between *K* and *d*, Fig. 19, we deduce at once that the two reciprocals must intersect in a common point, *e*;—for, on a comparison of the two figures, it will be seen that spaces in the structure correspond to points on the reciprocal diagram.

Passing next to the joint, *e f K*, we know by previous construction the reciprocal of, *e K*; and we have to find the reciprocals of, *e f* and *K f*;—in other terms, the points, *e* and *K*, Fig. 20, corresponding to the spaces, *e* and *K*, Fig. 19, are known; but the reciprocal point of space, *f*, has yet to be determined.

Therefore from points, *e* and *K*, Fig. 20, draw two lines parallel respectively to bars, *e f* and *K f*, and intersecting in, *f*, which is the reciprocal of the space similarly lettered on Fig. 19.

Proceeding next to nucleus, *d e f g c*, we know by previous construction the reciprocals of *d e*, *e f*, and *d c*; and we have therefore to find those of *f g* and *c g*. Hence, from the known

points, f and c, on the reciprocal diagram, draw lines parallel to these two bars meeting in, g, the letter common to their descriptions. The closed figure, $d\,e\,g\,f\,e\,d$, will form the reciprocal polygon of the nucleus similarly described. The rest of the figure is built up in a similar manner.

Another example of a roof truss, done on the letter system, is shewn in Figs. 21 and 22. The method of construction followed is the same as in the preceding example, and need not be described in detail, further than to remark that the line of forces is constructed by commencing from a point, Q, and laying off in succession the loads, $Q\,a$, $a\,b$, $b\,c$, $c\,d$, $d\,e$, $e\,f$, $f\,R$, returning to Q by the reactions, $R\,S$ and $S\,Q$. The greater part of this truss is in tension, the members of the upper arch alone being under compression.

The following solution of the truss, shewn in Fig. 23, was originally published in a modified form, in the pages of *Engineering*,* and a further development of it was pointed out to the author by Professor Main, of South Kensington and the Royal School of Mines. It is in many respects an improvement on the method given later in this work; though it is not so generally applicable and requires a certain amount of forethought and intuition.

Take the loads as given in Fig. 23; letter the spaces, and draw the line of forces, Fig. 24, for one half of the truss.

Beginning at nucleus, $N\,a\,l\,M$, draw from point, a, a line, $a\,l$, and from, M, a line, $M\,l$, intersecting in, l. The closed polygon, $N\,a\,l\,M\,N$, will be the reciprocal of the nucleus similarly described.

Proceeding next to joint, $ablk$, draw from point, b, a line parallel to, bk; and from, l, a line parallel to, lk, intersecting in, k, and forming the closed polygon, $alkb$, reciprocal of the corresponding nucleus.

Proceeding next in order to joint, $bkjihc$, we already know the reciprocals, bk and kj, but not those of the other three lines issuing from this joint. We have, however, previously fixed the point, c;—therefore from, c, draw an

* See a communication from Mr. H. G. Aubin, in the number of this journal for March 28th, 1879.

indefinite line, $c h_1$, and from the known point, j, a second indefinite line, $j i_1$.

Passing now to nucleus, $g\ i\ h\ f$, we shall find that we know none of the reciprocals of the four lines issuing from this joint. But it is easily foreseen that the reciprocal of, $h f$, must be contained between the indefinite lines, $c h_1$ and $d f_1$, being at the same time parallel to bar, $h f$; or, in other terms, we can find its magnitude and direction by drawing a line, $h_2 f_2$, parallel to, $h f$, between the limits of $c\ h_1$ and $d f_1$, Fig. 24. Resolve the stress, $h_2 f_2$, so found, in the two directions, $f g$ and $h\ i$, so as to form the triangle of forces, $h_2\ f_2\ i_2$. The line, $f_2 i_2$, represents the resultant tension at this joint, of the two stresses, $f\ g$ and $g\ i$.

As regards the reciprocal of the bar, $M g$, it will be readily granted that it must lie in the indefinite direction, $M\ g_1$, Fig. 24.

It will also be evident that the reciprocals of, $f g$ and $g\ i$, are necessarily situate on the same straight line; for the four lines of the reciprocal, $f\ g\ i\ h$, must form a closed quadrilateral, and it is literally impossible to form a closed figure composed of four lines, two of which are parallel and consecutive, unless these two parallel and consecutive lines are in one and the same direction.

Now the point, i, Fig. 24, must lie somewhere on the line, $j i$. It must likewise lie on a line, $i_2 i$, drawn through i_2, parallel to $c\ h$ or $d f$. Hence it will be found at the intersection, i, of the lines, $j\ i_1$ and $i_2\ i$.

If, therefore, the auxiliary triangle, $h_2\ f_2\ i_2$ be moved parallel to itself along the guiding lines, $c\ h_1$ and $d\ f_1$, the apex, i_2, will intersect the line, $j\ i_1$, in i, and simultaneously fix the three points, h, i, and f. But, since $f g$ and $g\ i$, are necesssarily in the same straight line; if $f\ i$ be produced, it will meet the indefinite line, Mg, in g, and fix that point. The reciprocal of the half truss is now completely drawn.

We shall return to the method of lettering in the chapter on Wind Pressures.

EXAMPLES.

1. Construct the reciprocal diagram of the roof structure, *Weymouth Goods-Shed,* Fig. 166, Plate I.; and find the natures and amounts of the stresses, produced in bars, x, y, z, by loads applied as given in the outline figure.

$x = -8{\cdot}7$ tons, (tension)
$y = +2{\cdot}9$ tons, (compression), $z = -5{\cdot}4$ tons.

2. Construct the reciprocal diagram of the roof structure *Didcot Provender Store,* Fig. 167, Plate I.; and find the natures and amounts of the stresses, produced in bars, x, y, z, by loads applied as shewn in the figure.

$x = +3{\cdot}5$ tons; $y = +15{\cdot}6$ tons; $z = -3{\cdot}3$ tons.

3. Construct the reciprocal diagram of the roof structure, Fig. 168, Plate I.; and find the natures of, and the ratio between the stresses, produced in the bars, x and y, by loads applied as shewn in the figure.

$$\frac{-x}{-y} = 1{\cdot}61$$

4. Construct the reciprocal diagram of the roof structure, Fig. 169, Pl. I.; and find the natures of, and the ratio between the stresses, produced in bars, x and y, by the loads given on the figure.

$$\frac{-x}{+y} = -1{\cdot}54$$

5. Find the natures of, and the ratio between the stresses, produced in bars, x and y, Fig. 170, Pl. I., by the graphically described loads.

$$\frac{+x}{-y} = -4{\cdot}375$$

6. Find the natures of, and the ratio between the stresses, produced in bars, x and y, Fig. 171, Pl. I., by loads applied as shewn on the figure.

$$\frac{-x}{+y} = -4$$

7. Find the natures of, and the ratio between the stresses, produced in bars, x and y, Fig. 172, Pl. I., by the given applied loads.

$$\frac{-x}{+y} = -1\cdot\dot{5}\dot{5}$$

8. Construct the reciprocal diagram of the *Neath Bridge* outside girder, Fig. 174, Plate II.; and find the natures and amounts of the stresses, produced in bars, x, y, z, by a series of static loads, each equal to 1·6 ton, applied at each of the apices, A, B, C, D, E.

$x = -$ 8·341 tons; $y = +$ 0·375 tons; $z = -$ 0·37 tons.

9. Find the natures and amounts of the stresses, produced in the same members of the *Neath Bridge*, by the action of the same static loads, with the addition of a rolling load of 5·6 ton, concentrated for the moment at joint, A.

$x = -$ 11·211 tons; $y = +$ 1·01 ton; $z = -$ 1 ton.

10. Determine the stresses, produced in the same members of the *Neath Bridge*, by virtue of the general static loads, and *two* rolling loads, each equal to 5·6 tons, concentrated for the moment at joints, A and B.

$x = -$ 16·95 tons; $y = +$ 2·32 tons; $z = -$ 2·28 tons.

11. Determine the stresses in the bars, x, y, z, of the same structure, under the influence of the general static loads, and *three* rolling loads, each equal to 5·6 tons, concentrated at the joints, A, B, and C.

$x = -$ 25·56 tons; $y = +$ 4·25 tons; $z = -$ 4·14 tons.

12. Find the maximum stress, produced in the division, x, of the lower boom of the *Neath Bridge* girder, in virtue of the general static loads, and a series of rolling loads, each equal to 5·6 tons, concentrated at the joints, A, B, C, D, E.

$x = -$ 37·5 tons.

Fig. 19.

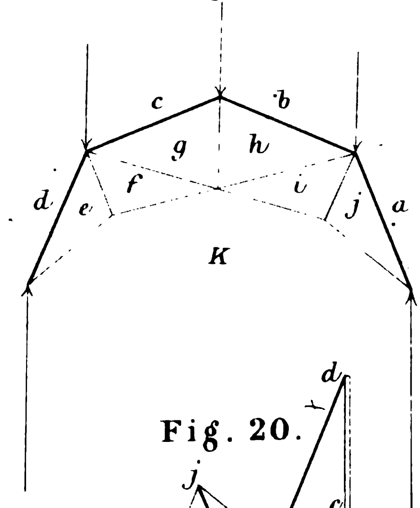


Fig. 21.

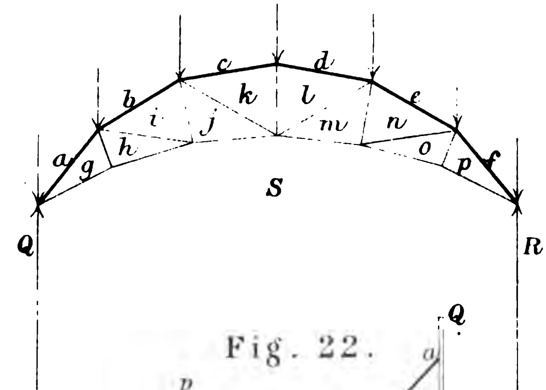


Fig. 20.

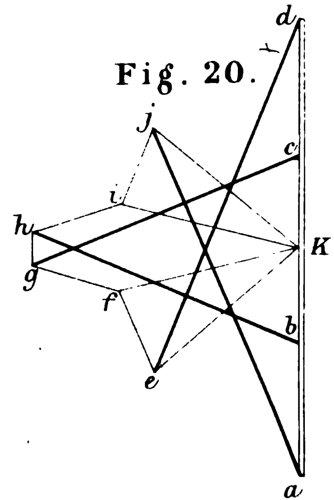
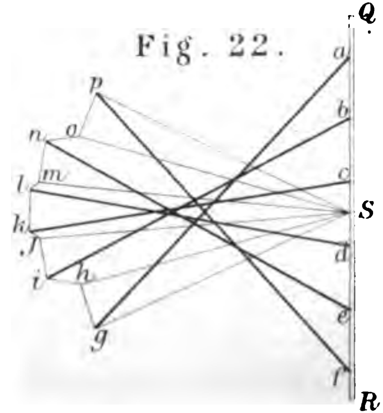


Fig. 22.



13. Construct the reciprocal diagram of the platform girder, *Cardiff Station*, Fig. 175, Pl. II.; and find the natures and amounts of the stresses, produced in bars, x, y, z, by loads applied as shewn in the figures.

$$x = +1\cdot5 \text{ ton}; \; y = +0 : z = +1\cdot5 \text{ ton}.$$

14. Determine the natures of, and the ratio between the stresses, produced in bars, x and y, of the braced bridge-structure, Fig. 176, Pl. II.

$$\frac{+x}{-y} = -1\cdot75$$

15. Determine the natures of, and the ratio between the stresses, produced in bars, x and y, of the bridge-structure, Fig. 177, Pl. II.

$$\frac{-x}{+y} = -1\cdot58$$

16. Find the natures of, and the ratio between the stresses, produced in bars, x and y, of the bridge-span, Fig. 178, Pl. II.

$$\frac{+x}{+y} = 2\cdot6$$

17. Compare the natures and ratio of the stresses in bars, x and y, of the braced cantilever, Fig. 179, Pl. II.

$$\frac{x}{y} = -\infty$$

PART II.

COMBINED ANALYTIC AND GRAPHIC METHODS.

THE weight of an individual member belonging to any structure is a vertical force acting through the centre of gravity of the member, which admits of being resolved into component parts applied at other centres of force. The principle by which this change is effected may be expressed in the following form:

If three balanced parallel forces act together in the same plane, each of those forces will be proportional to the distance between the lines of action of the other two forces.

By the aid of this principle all weight-forces can be reduced to equivalent components acting through other centres of force. Subsequently these component parts can be compounded with such external forces as have the same points of application.

The principle just explained is often needed in the operations of graphic statics, and especially when it is necessary to take into account the dead weights of beams, roofs, and bridge-platforms.

1. BEAM UNDER INCLINED STRESS—In Fig. 25, S_1 S_2, is a beam connecting the fixed points, S_1 and S_2, and supporting a vertical load directed in the path marked, 1.

To find the stresses in this case, construct the polygon of forces, 1, 2, 3, Fig. 26, corresponding to the vertical force, 1, and the resistances along the directions, 2 and 3, Fig. 25.

It will be seen that the polygon of forces is the reciprocal of the nucleus, *D*, and therefore gives the values of the stresses applied, or supposed to be applied, at that point.

Fig. 23.

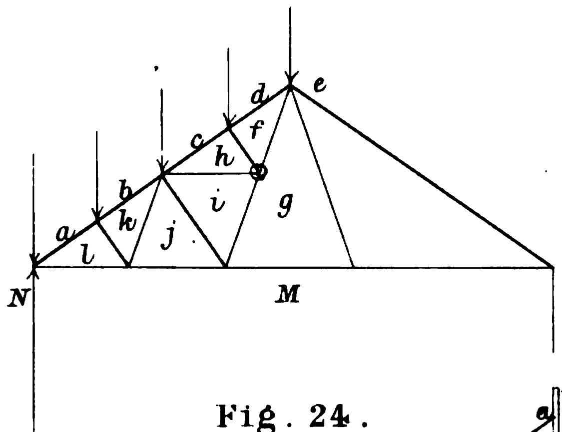
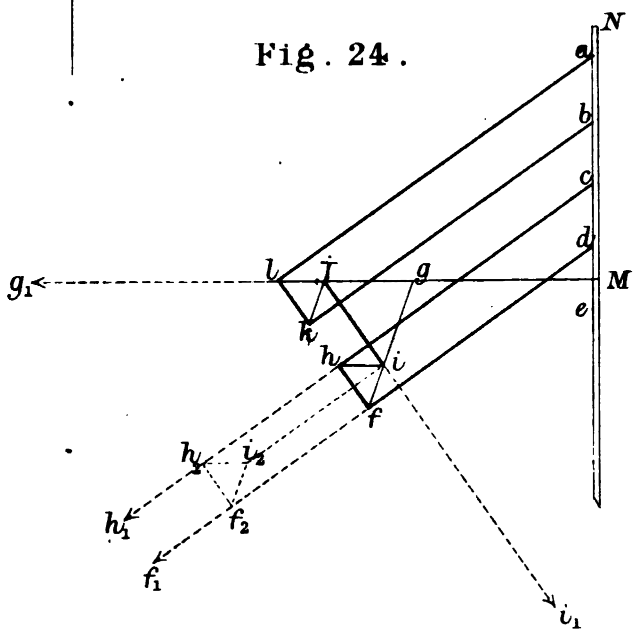


Fig. 24.



Secondly, construct the polar polygon relatively to the pole, *O*. This polygon, (1′ 2′ 3′), will be found to close; so that in this instance both the polar polygon and the polygon of forces form closed figures. That such should be the case is evident;—1°; because the forces are in equilibrium, and 2°; it will be seen that the points, 1′ and 3′, are determined by drawing a line, 3′ 1′, across the force-lines, 1 and 3, parallel to the polar line, $O_{1\,3}$; and further the point, 2′, is subsequently found by drawing a line, 1′ 2′, parallel to polar line, $O_{1\,2}$. It remains to draw the line, 2′ 3′, parallel to polar line, $O_{2\,3}$. Now we know that the extreme lines, 3′ 1′, and, 2′ 3′, must intersect on the resultant line of action of the forces, 1 and 2, (Pt. I. Ch. I. § 3). But force-line, 3, indicates the resultant objective path of forces, 1 and 2. Hence the lines, 3′ 1′, and, 2′ 3′, must meet in the point, 3′; and the polar polygon will form a closed figure (1′ 2′ 3′).

Fig. 25

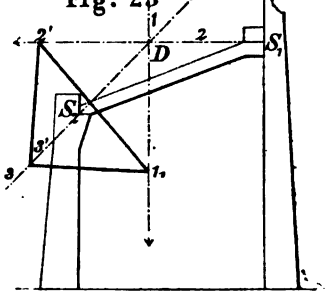
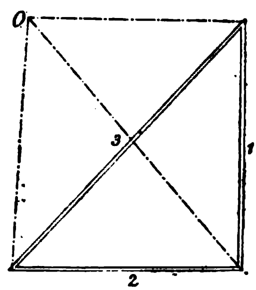


Fig. 26.



From the preceding demonstration is deduced the following general rule:

When a system of forces is in equilibrium, two events will happen;—1°; the polygon of forces will form a closed figure; 2°; the polar polygon will simultaneously close. These are the necessary and sufficient conditions for equilibrium, and one is not sufficient without the other.

PROBLEM.—*Given* (*Fig.* 27) *the vertical load*, 1, *in magnitude and position, the line of action*, 2, *of one of the resistances, and the centre*, S_2, *through which the third force or resistance*, 3,

passes ;—required the direction of force-line, 3, *and the magnitudes of the two resistances,* 2 *and* 3.

Graphic Solution.—Set out the given lines of action, 1 and 2, (Fig. 27) and (Fig. 28), the vertical line, $\overline{AB}$, graphically representing the given magnitude of the force, 1. From point, B, draw the indefinite line, $\overline{BX}$, parallel to the given direction of force, 2.

In the plane of the forces pitch any pole, O_1, from which draw the polar lines, $\overline{O_1 A}$ and $\overline{O_1 B}$, to the points, A and B.

Setting out from, S_2, draw the polar polygon relatively to the pole, O_1 ;—that is, draw a line, $S_2\,1'$, parallel to the line, $\overline{O_1 A}$, meeting force line, 1, in 1′. From, 1′, draw a line, 1′ 2′, parallel to line, $\overline{O_1 B}$, meeting force-line, 2, in 2′.

Join the points, S_2 and 2′, by a line, $S_2\,2'$, and from the pole, O_1, draw a line, $\overline{O_1 C}$, parallel to, $S_2\,2'$, meeting line, $\overline{BX}$, in a point, C.

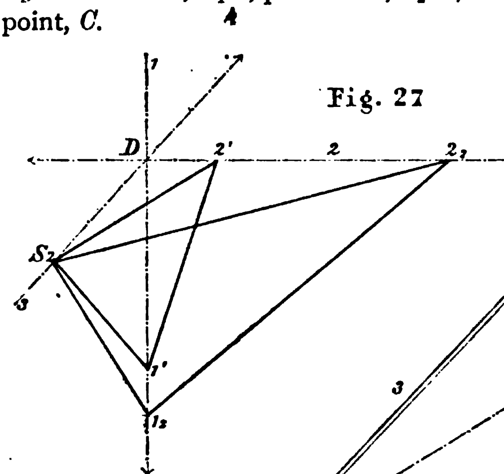


Fig. 27

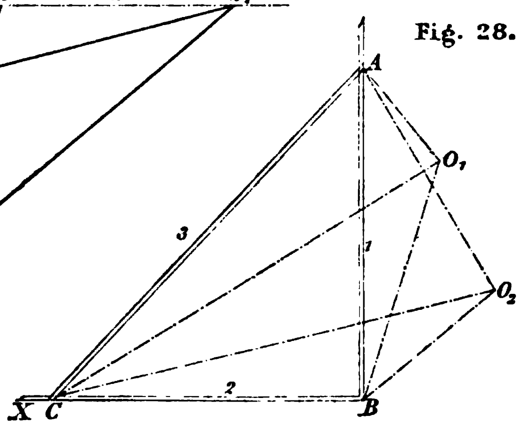


Fig. 28.

The line, $\overline{BC}$, so determined, will be the graphic representation of the magnitude of force, 2, and, $\overline{CA}$, will give the *value* of the resistance, 3.

Lastly, from centre, S_2, draw a line, $\overline{S_2 D}$, parallel to, $\overline{AC}$, which will define the *direction* of the third force.

Moreover, in this particular case, it will be found that the point, D, forms the local junction of the three force-lines, and this fact suggests a much shorter graphic process, which consists in simply joining, S_2 and D, by a line, $\overline{S_2 D}$, and then drawing its reciprocal, $\overline{A\,C}$, Fig. 28, which will meet the line, $\overline{B\,X}$, in C, and complete the polygon of forces.

As a check upon the graphic process, a second polar polygon has been constructed relatively to an independent pole, O_2. Both polygons agree in the determination of the same point, C, on the polygon of forces.

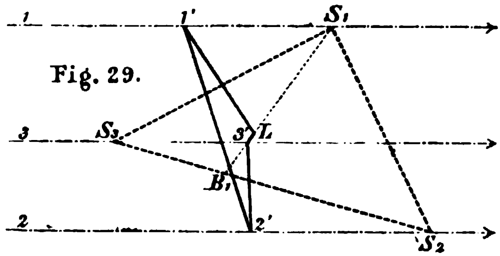
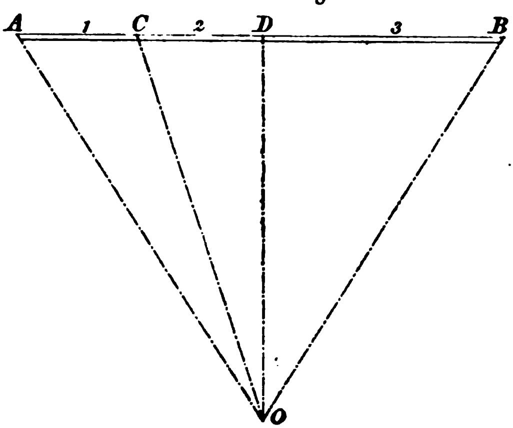


Fig. 29.

Fig. 30.



2. Resultant of Parallel Forces.—*To determine the magnitude and direction of the resultant of three parallel forces applied to a body.*

Let three parallel forces act through the centres, S_1, S_2, and S_3, Fig. 29, and let them be supposed to be turned down or

rebatted on the plane of the paper. For example, let the points, S_1, S_2, and S_3, represent the heads of three piles; let a sheet of paper be laid across the piles, and let the applied parallel forces, whether vertical or inclined, be turned down through an angle on to the plane of the paper; so that their

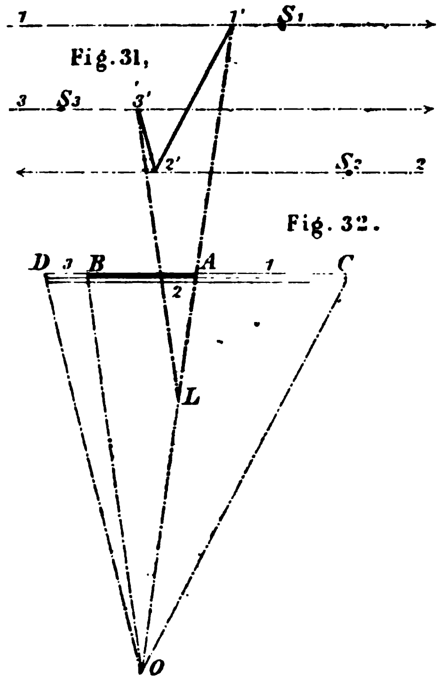


Fig. 31,
Fig. 32.

directions take the sense and position indicated by the arrows in Fig. 29.

Construct the polygon or line of forces along a line, $\overline{AB}$, Fig. 30, equal to the sum of the forces, 1, 2, 3.

Pitch a pole, O, and draw the polar lines, $\overline{OA}$, O_{12}, O_{23}, and $\overline{OB}$.

In order to find a point on the resultant path of the fourth force, draw across force-line, 1, a line, $1'L$, parallel to, $\overline{OA}$, meeting force-line, 1, in, $1'$. Complete the polar polygon by drawing lines, $1'2'$, $2'3'$, $3'L$, parallel respectively to lines, O_{12}, O_{23}, and $\overline{OB}$.

It will be found that the first and last lines of this polar polygon, viz., lines, $3' L$ and $1' L$, meet in a point, L, which consequently lies on the line of action of the resultant force (Pt. I. Ch. I. § 3). Through, L, draw a force-line, 4, parallel to the other force-lines, and equal in length to the sum of the forces, (1, 2, 3), represented by line, $\overline{A\,B}$. The force-line, 4, will express the graphic value of the resultant force of the given system.

The construction for the similar case, when certain of the forces are negative, is given in Figs. 31 and 32, in which force, 2, is taken in a negative direction. The graphic operation is the same, save that the sense of the line, 2, on the polygon or line of forces, is reversed. The line of forces is drawn by commencing as in the former case at the origin, A, and setting off the part, $\overline{A\,C}$, to the right, $\overline{C\,D}$, to the left, and lastly, $\overline{D\,B}$, to the right. The part, $\overline{A\,B}$, will represent the algebraic sum of the forces, or the magnitude of the resultant force.

It may be well to add that if the points, A and B, coincide, the line of forces closes; but the polar polygon, ($1'$, $2'$, $3'$, L,) remains open; since in that case the line, $3' L$, would lie parallel to the line, $1' L$; and the centre, L, would be placed at an infinite distance from, $3'$;—that is, there would be no point discoverable on the path of the resultant, or the forces would be reduced to a couple. This is, moreover, evident, since under those conditions the sum of the two positive forces, 1 and 3, would equal in amount the negative force, 2, and the system would be reduced to a couple, having the value,

$$F \times d$$

in which expression, F, is the sum of the forces, 1 and 3, and, d, is the distance between the line of action of the resultant, F, and the force-line, 2.

The above remarks corroborate the statement made with respect to the conditions necessary for equilibrium (Part II. § 1).

By the aid of the preceding theorem it can easily be determined what proportion of the incumbent weight is independently borne by each of three supporting props, given in

position. For, taking the diagram, Fig. 29, suppose that the resultant load acting through, L, were given, and it were required to find the partial loads brought to bear on each of the three props, S_1, S_2, and S_3; the common directions of the forces being rebatted on to the plane of the paper.

Fig. 33.

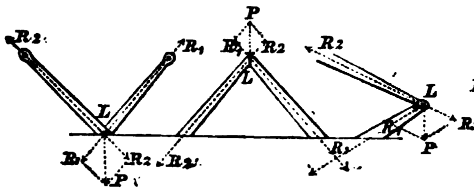
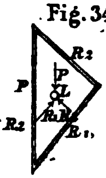


Fig. 34.



The solution of this problem takes the following shape:—

1°. Draw a line, $\overline{AB}$, representing the given load.

2°. Pitch a pole, O, and draw the polar lines, $\overline{OA}$ and $\overline{OB}$.

3°. Commencing at the given point, L, Fig. 29, draw the polar polygon relatively to the pole, O;—that is to say, draw across force-line, 1, a line, $L\,1'$, parallel to polar line, $\overline{OA}$, and

Fig. 35.

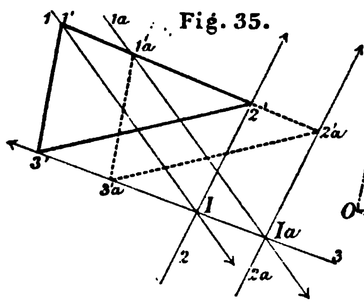
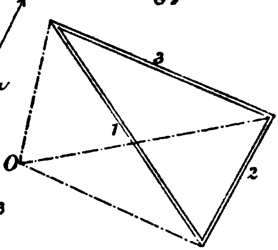


Fig. 36.



from the same point a second polar line, $L\,3'$, parallel to $\overline{OB}$, meeting force-line, 3, in a point, 3′.

Next, connect the three centres, S_1, S_2, and S_3, by lines forming a triangle, and through, L, draw a line, $\overline{S_1L}$, meeting the line, $\overline{S_2S_3}$, at, B_1.

Now, since the system is in equilibrium, the resultant of the forces, 2 and 3, will pass through point, B_1; otherwise this partial resultant would not act in the same plane with the

forces acting through, S_1 and L, and could not form with them a system of balanced forces.

Let, therefore, the forces, 2 and 3, be replaced by a resultant force equal to their sum and passing through, B_1.

We have now to deal with three parallel forces, viz.; L, the load through the point, L; R, the sum of the forces, 2 and 3, applied at B_1; and force, 1, acting at, S_1. Each of these forces is proportionate to the distance between the lines of action of the other two; or symbolically,

$$L : R :: S_1B_1 : S_1L$$

wherefore,

$$\frac{L}{R} = \frac{S_1B_1}{S_1L};$$

which ratio clearly shews that the resultant total load acting at, L, bears the same proportion to the sum of the forces, 2 and 3, that the line, $\overline{S_1B_1}$, does to the line $\overline{S_1L}$. But in Fig. 30 the line $\overline{A\,B}$, graphically represents the resultant load, acting at, L;—whence it follows that if a part, $\overline{B\,C}$, be marked off on this line in such a manner that

$$\frac{A\,B}{B\,C} = \frac{S_1\,B_1}{S_1\,L};$$

the point, C, so determined, will divide the line, $\overline{A\,B}$, into parts, one of which, $\overline{B\,C}$, will be equal to the sum of the forces, 2 and 3; and the remaining part, $\overline{A\,C}$, will represent the force, 1, at S_1.

Having thus found the point, C, draw the polar line, $\overline{O\,C}$, or $O_{1\,2}$, and from point, 1′, previously fixed, draw a line, 1′2′, parallel to the polar line, $O_{1\,2}$, just established. Join 2′ and 3′, and draw polar line, $O_{2\,3}$ or $\overline{O\,D}$, parallel to 2′ 3′ so found.

The whole line, $\overline{A\,B}$, will then be divided by means of the polar lines, $\overline{O\,C}$ and $\overline{O\,D}$, into three parts (1, 2, 3) proportionate to the separate loads bearing on the three props, S_1, S_2, and S_3.

3. FRAME COMPOSED OF TWO BARS.—Fig. 33, represents frames made up of two bars, loaded at the joint, L, with a

given force, P, and held up by the resistances, R_1 and R_2, acting directly along the bars.

The polygon of forces is constructed in the usual way, and the nature of the stresses is determined in each case by the customary rule for tensions and compressions. Case I. has been worked out and the details of the construction are given in Fig. 34, from which it will be seen that both the stresses, R_1 and R_2, lie outside their respective bars, and represent tensions. The converse holds in Case II., and in Case III., R_2 is a tensional, and R_1 a compressive force.

4. TRIANGULAR FRAME.—Let, Fig. 35, represent a triangular frame, having its apices, 1′, 2′, and 3′, on the force-lines, 1, 2, and 3, along which are applied three forces, holding the frame in equilibrium.

Here we have an example of the base-triangular figure becoming subject to the laws of Graphic Statics, forming an exception to the general rule given in Part I. Chap. I. § 5. This anomaly is accounted for by the presence of a force-line at each apex, which completes the number of three lines issuing from each of those points, and therefore brings the figure under ordinary rules.

Graphic Solution.—To find the stresses induced in the bars, construct first the polygon of forces, (1, 2, 3,), having its lines respectively parallel to the three given force-lines, 1, 2, and 3.

Next, construct the reciprocal of nucleus, 1′, which, according to the general rule, will give the stresses along the bars, 1′ 3′ and 1′ 2′, intersecting at that point.

There being three lines diverging from point, 1′, its reciprocal will form a triangle. Draw, therefore, the line, $O_{1\,3}$, parallel to 1′ 3′; and line, $O_{1\,2}$, parallel to, 1′ 2′. Let these lines intersect in a point, O. The triangle formed by these two polar lines, $O_{1\,2}$ and $O_{1\,3}$, and the side, 1, of the polygon of forces, will constitute the reciprocal of the point, 1′.

Proceeding to nucleus, 2′, we already know the reciprocal, $O_{1\,2}$, of the bar, 2′ 1′. We also possess the reciprocal of force-line, 2, which is furnished by the triangle of forces. It only remains, therefore, to draw the reciprocal of bar, 3′ 2′. Now, the line, 3′ 2′, forms a closed figure with force-lines, 2 and 3;

—wherefore, from the junction of forces, 2 and 3, on the polygon of forces, draw a line, $O_{2\,3}$, parallel to bar, 3′ 2′. This line must necessarily enter the pole, O; for the reason that the lines reciprocal of the closed triangular frame, (1′ 2′ 3′), must intersect in one point and form a nucleus, O. In fact, the triangular frame is simply the polar polygon of the given system of applied forces, relatively to the pole, O.

In connexion with this case, it is well to note that for the same pole there can exist only one class of triangular frame in equilibrium under the action of three forces, the direction and magnitude of which are given. This statement is sufficiently proved by considering the fact that any apex of the frame, for example that marked, 2′, must fulfil two conditions :—

1°. It must lie at the intersection of lines, 1′ 2′ and 3′ 2′.

2°. It must simultaneously be situate on a line, drawn parallel to force, 2, and passing through a point, I, at which force-lines, 1 and 3, intersect.

The second condition is a consequence of the balanced state of the forces; and in truth the frame is only in equilibrium because the forces acting upon it, are independently in equilibrium.

Consequently, if we suppose the force, 2, to suffer a slight displacement, by which it is shifted to a position, 2_a, Fig. 35; no triangular frame can be constructed in equilibrium under the action of the forces, 1, 2_a, and 3. In order to make a triangular frame possible, we should have to suppose the bar, 3′ 1′, shifted parallel to itself into the position, $3'_a\,1'_a$, which would involve a change of position of the force-line, 1, from 1 to 1_a.

It can now be shewn that the three forces, 1_a, 2_a, and 3, meet in a common point, I_a, and constitute an equilibrated system. On the hypothesis of the force, 2, being independently displaced, we may suppose two forces, each equal to 2_a, added to the system and made to act in opposite directions through the point, I. The addition of these two balanced forces will not disturb the state of stress induced by the three forces, 1, 2_a, and 3. Now, one of the added forces would take the place of the transferred force, 2, which formerly acted

through, I; the remaining force, 2_a, would in that case constitute a couple with the transposed force, 2_a, tending to turn round the frame from right to left. Equilibrium could only be re-established by means of the displacement of the force, 1, from 1 to 1_a, as previously stated and explained.

5. TRIANGULAR FRAME UNDER PARALLEL FORCES.—A particular case of the preceding problem consists in supposing

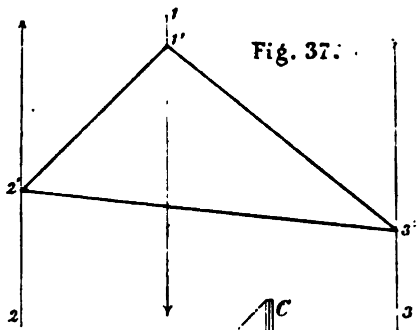


Fig. 37.

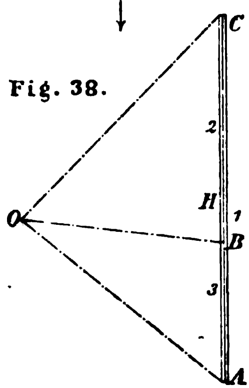


Fig. 38.

the frame to be subject only to parallel forces. The construction for this case may be briefly summarised as follows :—

1°. Given the parallel force-lines, 1, 2, and 3, applied at the joints, 1′, 2′, and 3′, Fig. 37, draw the line of forces, $\overline{CA}$, representing the whole vertical or inclined load.

2°. Recalling to mind that the triangle, 1′ 2′ 3′, is the polar polygon of the system of forces, find the pole, O, by drawing polar line, $\overline{OC}$, parallel to, 1′ 2′, and polar line, $\overline{OA}$, parallel to 1′ 3′. The intersection of these two lines will determine the pole required, from which draw the third polar line, $\overline{OB}$, or

$O_{2,3}$, parallel to the bar, 2′ 3′. This line will cut the line of loads in a point, B, dividing it in the ratio of the reactions at, 2′ and 3′.

To find the nature of the stresses, let us examine the reciprocal of nucleus, 1′, which corresponds to the triangle, $\overline{OCA}$. Commencing with the known direction of the force, 1, we obtain the directions of the stresses along bars, 1′ 2′ and 1′ 3′, by passing round the triangle, $\overline{OCA}$, in the order of the lines, $\overline{CA}$, $\overline{AO}$, $\overline{OC}$; and imagining the directions thus determined to be applied at and approaching the nucleus, 1′, we learn that the stresses are both compressive, since they lie along the bars subject to their actions.

On the other hand, on an examination of the triangle, $\overline{OBC}$, which is the reciprocal of nucleus, 2′, commencing with the known upward direction of reaction, 2, we perceive that the stress along bar, 2′ 3′, is tensional.

In this instance the forces and the frame, to which they are applied, will be in equilibrium, notwithstanding that the forces suffer displacement; for an indefinite number of triangular frames can be erected in stable equilibrium under the action of three parallel forces. The only modification in the different cases will take place in the distribution of the stresses.

If the bar, 2′ 3′, were horizontal instead of being inclined, its reciprocal, $\overline{OB}$, would become horizontal. But, at the same time, one of the other bars and its reciprocal would suffer displacement. It may, however, be stated that an ideal horizontal bar can be supposed to exist in the frame, the stress along which would be equal in amount to the perpendicular distance, OH, Fig. 38.

6. POLYGONAL FRAME.—Let the frame (5′ 1′ 2′ 3′ 4′), Fig. 39, be supposed subject to five forces, 1, 2, 3, 4, and 5, acting through the joints, 1′, 2′, 3′, 4′, and 5′, of the polygonal frame. It is required to find the stresses induced by the given forces, 1, 2, and 3, and the end reactions, 4 and 5.

Graphic Solution. Having drawn the skeleton-outline of the frame, and the objective paths of the forces, and reactions, construct as usual the polygon of forces. Beginning at the

point, A, draw in succession the lines, 1, 2, and 3, representing in magnitude and direction the three given forces.

From A, draw an indefinite line, $\overline{A\,B}$, parallel to the end reaction, 5, and similarly from the other end, D, a line $\overline{D\,B}$, parallel to the end reaction, 4. These two lines will intersect in, B, and complete the polygon of forces of the given system.

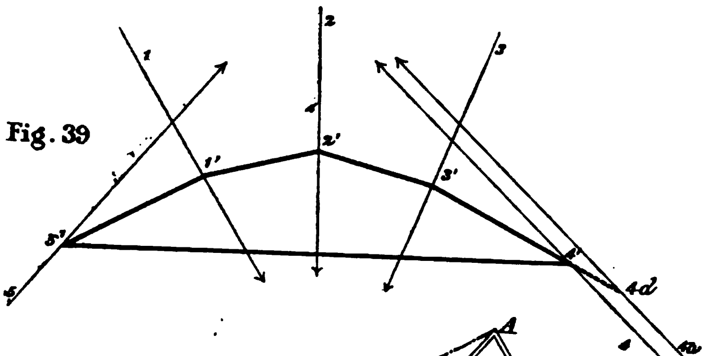


Fig. 39

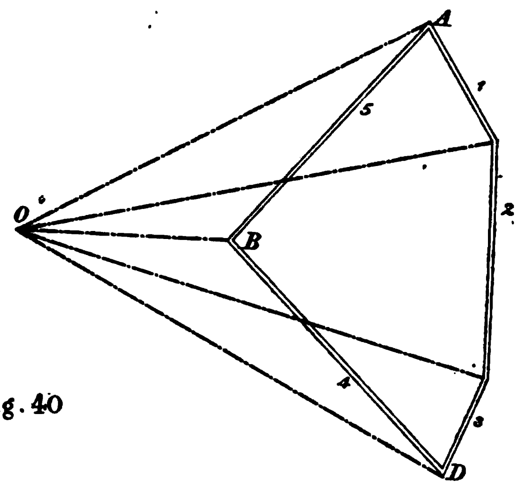


Fig. 40

Granting that the frame is the polar polygon of the system, draw a line, $O_{1\,5}$, parallel to bar, 1′, 5′; and a second line, $O_{3\,4}$, parallel to the bar, 3′ 4′. These two lines will meet at the pole, O, from which draw polar lines to the several nuclei on the polygon of forces.

It can be now shewn that the triangle, formed of the lines, $O_{1\,5}$, force 1, and $O_{1\,2}$, is the reciprocal of nucleus, 1, so that the

lines, $O_{1\,5}$ and $O_{1\,2}$, graphically represent the stresses along the bars, 1′ 5′ and 1′ 2′. Similarly, $O_{2\,3}$, gives the stress along bar, 2′ 3′; $O_{3\,4}$, that along 3′ 4′, and so on.

As in the previous example of a triangular frame, so here again there exists for the same pole, only one class of frame in equilibrium under the given system of forces. If, therefore, any of the force-lines, such as 4, were to suffer a slight displacement, from 4 to $4a$, the frame would cease to be in equilibrium, unless other changes were subsequently made.

The stresses along the bars of the frame are all compressive with the exception of that along, 4′ 5′, which is tensional. Were the bar, 4′ 5′, omitted, it would be necessary to suppress its reciprocal, O_{45}. The lines, 4 and 5, would then express the reactions along force-lines, 4 and 5, induced by the action of the other forces. Rankine (*Applied Mechanics*, § 151) is not consistent in his treatment of this case; for, when dealing with the open frame, he suddenly changes the conditions of the general problem, and supposes the reactions to take place directly along the extreme bars, 3′ 4′ and 1′ 5′; whilst in the general case of a closed frame he allows the reactions, 4 and 5, to be directed in any paths traversing the centres, 4′ and 5′.

A pentagonal frame, as such, is not subject to the laws of Graphic Statics. The reason is that the number of its sides and nuclei does not satisfy the general equation,

$$S = 2\,p - 3$$

In fact a pentagonal frame is not in stable equilibrium, unless two additional bars be introduced. For example, it will be seen that if in the present case the joints are loose and unstayed, the stresses, being all compressive, will bend the frame out of shape, unless accessory bars be added to preserve the stability of the joints. Two such bars, connecting joint, 1′, with joint, 4′, and, 3′ with 5′; or again two stays connecting the ridge-point, 1′, with the ends, 4′ and 5′, would equally well answer the purpose. The general formula, viz.,

$$S = 2\,p - 3$$

is then satisfied and expresses an identity.

If the frame were inverted, the values of the stresses would still be given by the same diagram; but their natures would be interchanged. For example, tension along bar, 4′ 5′, would become compression, and the other stresses, which in the first instance were all compressive, would now be all tensional.

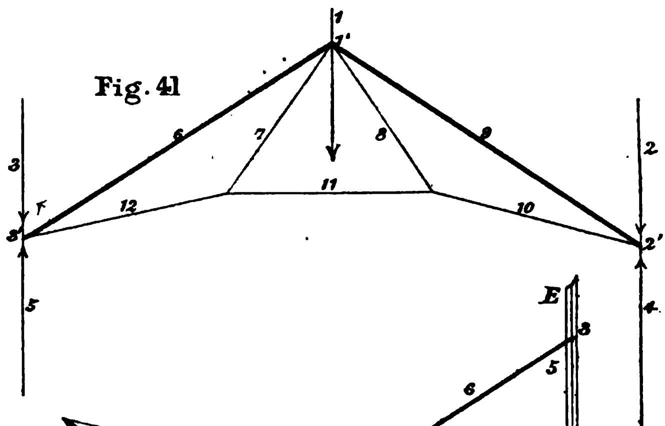


Fig. 41

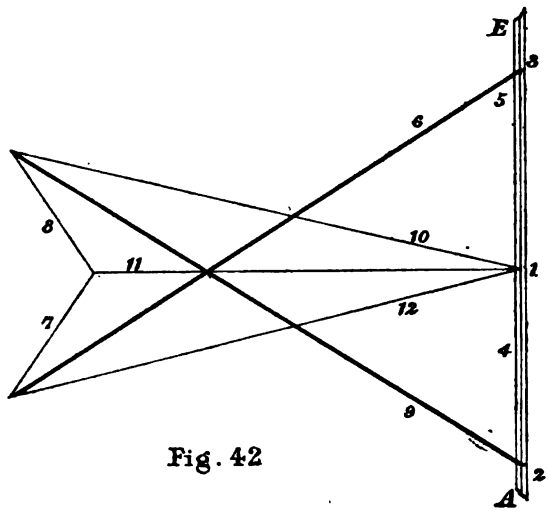


Fig. 42

In that case the frame would exist in a state of internal equilibrium without the addition of accessory members; it could, nevertheless, oscillate about its central position, and in that sense it would not exist in what may be termed *static equilibrium.*

7. ROOF FRAMES. FIRST EXAMPLE.—Let Fig. 41, represent a roof frame, consisting of two rafters, 6 and 9; three

Fig. 43.

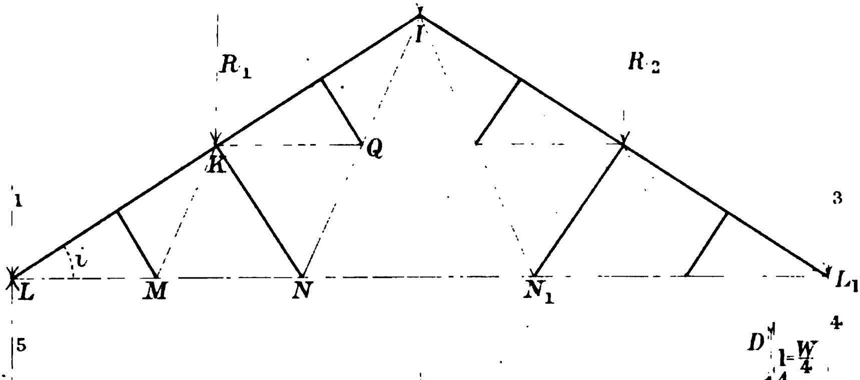


Fig. 44.

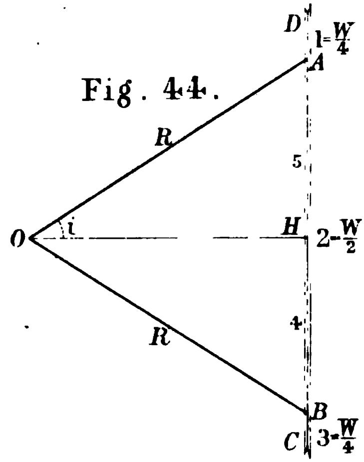


Fig. 45.

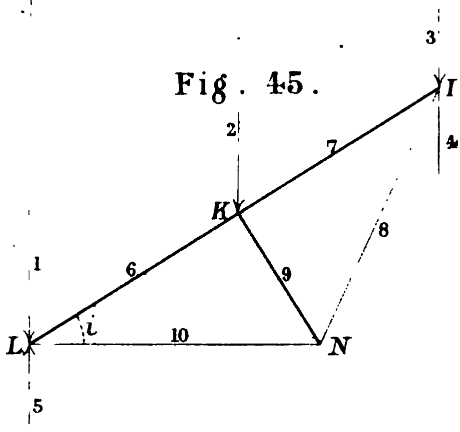


Fig. 46.

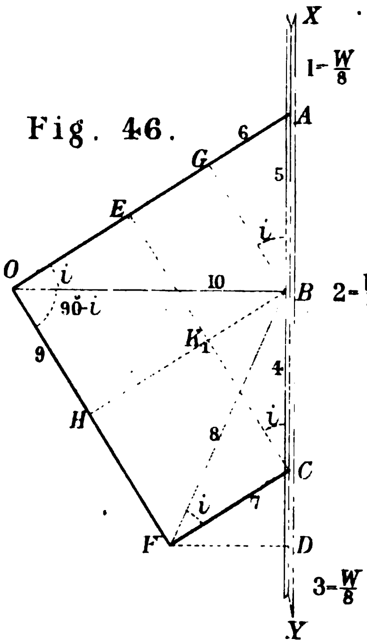
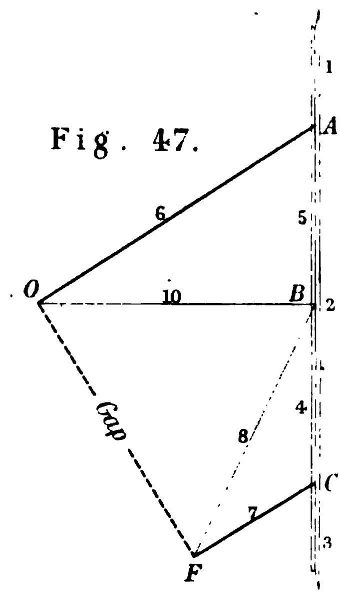


Fig. 47.



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tie-bars, 10, 11, and 12; and two accessory tie-bars, 7 and 8. A frame so constituted satisfies the equational criterion,

$$S = 2\,p - 3,$$

in which expression it is necessary to give, S and p, representing the number of sides and angular points, the values, 7 and 5, respectively.

Graphic determination of the Stresses.—The construction of the reciprocal figure, which graphically represents the various stresses, induced in the bars of the frame, by a vertical load acting through the ridge-joint, 1′, takes the following form :—

1°. Draw the skeleton-outline of the frame, Fig. 41, and the lines, 1, 2, 3, indicating the objective paths of the load and the distributed dead weight of the frame itself. The second part of the load, that is, the dead weight of the frame, is supposed to be separated into component parts acting through the ridge-joint, 1′, and the reaction-joints, 2′ and 3′, according to a principle which has been explained at the beginning of this part, and which will be developed more at length in the next example.

2°. Draw the load-line, $\overline{EA}$, composed of forces, 1, 2, 3; and, retracing the same line in an upward sense, divide it into halves, 4 and 5, representing the reactions, due to the half-loads, at 2′ and 3′.

3°. Commencing at the nucleus, (3, 5, 6, 12), construct its reciprocal by drawing from junction, (3 × 1), a line, 6, parallel to bar, 6; and from junction (4 × 5) a line, 12, parallel to bar 12. Similarly from, (2 × 1), draw a line, 9, parallel to bar, 9; and from (4 × 5) a line, 10, parallel to bar, 10.

Proceeding next in order to nucleus, (8, 10, 11), draw from junction (9 × 10) a parallel to bar, 8; and from (4 × 5), a line parallel to bar, 11. Passing now to the nucleus, (7, 11, 12), we know the reciprocals of all the bars meeting at this point, with the exception of that of bar, 7. Moreover, we know that the reciprocal of, 7, must pass through the junctions, (6 × 12) and (8 × 11), already determined. Therefore it follows, as a check upon the graphic process, that a line, 7, joining points, (6 × 12) and (8 × 11) must be parallel to bar, 7.

With regard to this particular kind of roof frame, Rankine remarks (*Applied Mechanics*, § 155):—

"If the distribution of the loads on the joints of a polygonal frame, though consistent with its equilibrium as a whole, be not consistent with the equilibrium of each *bar*, then, in the diagram of forces, when converging lines respectively parallel to the lines of resistance are drawn from the angles of the polygon of external forces, those converging lines, instead of meeting at one point, will be found to have gaps between them. The lines necessary to fill up those gaps will indicate the *forces* to be supplied by means of the resistance of braces."

This rule, which appears to be due to the late Professor Clerk Maxwell, is a very suggestive one, and is applicable in the case we have been considering; since if the braces, 7 and 8, were omitted, there would exist two gaps, between the junctions, (9×10); (6×12); and the horizontal line, 11. We have underlined two expressions in the extract, which seem to us to be incorrectly applied. Instead of the first term, *bar*, it would be better to read *joint*, and instead of the second term, *forces*, it would be more correct to insert, *stresses*.

8. Roof Frames. Second Example.—Let, Fig. 43, represent a roof frame, consisting of two side rafters, IL and IL_1; a tie-bar, LL_1, between which and the side rafters is inserted a series of secondary trusses as shewn in the figure. This frame-work will be found to satisfy the general equation of criterion;

$$S = 2p - 3,$$

in which expression, $S = 27$, and $p = 15$.

The present example confirms the statement made in the first part of this work, to the effect that the triangle is the *base-figure* of all graphic frames; for, the design is made up of a general triangular truss, $L I L_1$, which is divided into three secondary triangular trusses, $I N L$, $I N N_1$, and $I N_1 L_1$. Each secondary truss, $I N L$, is subdivided by means of a strut-brace, $\overline{KN}$, into two part-trusses, $K N I$ and $K N L$, and

these again are divided by the braces, $K M$ and $K Q$, into triangular parts.

Graphic Determination of the Stresses.—The necessary operations for constructing the figure, reciprocal of the given skeleton roof-frame, might be built up directly as in the last example, but it may be better to adopt an indirect method, which may be briefly summarised as follows :—

1°. Consider the primary truss, $I L L_1$, isolated from the minor truss-forms, and examine the distribution of the loads on the supposition that the primary truss supports, independently of the others, the whole weight of the roof. In this case, we have to deal with a triangular frame, $I L L_1$, loaded on each of its sides, $I L$ and $I L_1$, with half the weight of the whole roof ($= \frac{1}{2} W$). This weight, ($\frac{1}{2} W$), may be supposed to act at the centre of gravity of each side rafter, and can therefore be resolved into two component, and equivalent parts, $\frac{1}{4} W$ and $\frac{1}{4} W$, acting through the end, L, and ridge-joint, I. For instance, the half-weight, $R_1 = \frac{1}{2} W$, Fig. 43, directed through the centre, K, may be decomposed into two parts, each equal to $\frac{1}{4} W$, acting through, I and L. Similarly, $R_2 = \frac{1}{2} W$, will be divided into two parts, each equal to $\frac{1}{4} W$, having their points of application at, I and L_1, respectively.

Consequently the whole load is equivalent to a force, $\frac{1}{2} W$, applied at, I, and two forces, $\frac{1}{4} W$ and $\frac{1}{4} W$, acting through the ends of the rafters, marked, L and L_1.

We shall suppose that the part-load, $\frac{1}{2} W$, applied at, I, is the only force, which induces *thrust* along the bars, $I L$ and $I L_1$; since the remaining part-loads, $\frac{1}{4} W$ and $\frac{1}{4} W$, acting downwards through, L and L_1, would, if allowed to have free play, bring a *pull* to bear upon the same bars. But this tensional effect is removed by the resistances offered by the pillars at, L and L_1. Hence, it may be inferred that the pull along the tie-bar, $L L_1$, is due to the action of the side-thrusts along the rafters caused by a load, $\frac{1}{2} W$, concentrated at the ridge-joint, I.

Primary Truss, Graphic Diagram.—In order to construct the reciprocal figure of the primary truss, draw first the line

of forces, $\overline{DC}$, Fig. 44, on which, $\overline{DA}$ or force, 1, at L is made equal to $\frac{1}{4}W$; $\overline{AB}$ or force, 2, $= \frac{1}{2}W$; and $\overline{BC}$ or force, 3, at $L_1 = \frac{1}{4}W$. Retracing the same line make $\overline{DH} = \overline{CH} = \frac{1}{2}W$, which represent the amounts of the reactions at the pillars, and complete the line of forces.

Next, from junctions, (1×2) and (2×3), draw lines, R and R, parallel to the side-rafters, and meeting in the pole, O. If from the pole, O, a line, $\overline{OH}$, were now drawn, parallel to, LL_1, the point, H, so determined, would divide the line of loads in the ratio of the reactions at the pillars. In this case the reactions are equal and equivalent to half the total incumbent weight of the roof, which has been shewn to be true, independently of the reciprocal figure.

Primary Truss. Analytical Expressions for Stresses.—Having found the graphic values of the stresses, it is easy to deduce their analytical forms in relation to the primary truss.

Let, i°, be the angle of inclination of the side-rafters to the horizontal line, LL_1; then it follows that

$$\frac{AH}{OH} = tan.\, i\,;$$

or,

$$\overline{AH} = \overline{OH}\, tan.\, i\,;$$

but,

$$\overline{AB} = 2\,\overline{AH}\,;$$

therefore,

$$\overline{AB} = 2\,\overline{OH}\, tan.\, i\,;$$

that is,

$$\tfrac{1}{2}W = 2\,\overline{OH}\, tan.\, i\,;$$

and,

$$\overline{OH} = \tfrac{1}{4}W \div tan.\, i = H. \qquad (1.)$$

Again,

$$\overline{AO} = \overline{OH}\, sec.\, i = R\,;$$

therefore, by equation (1)

$$R = H\, sec.\, i = \tfrac{1}{4}W.\,\frac{1}{sin.\, i}. \qquad (2.)$$

The stresses due to the primary truss, independently considered, take the forms given in equations, (1) and (2); namely;—

Horizontal pulls along the tie-bar, $L L_1$, being symbolised by the letter, H, take the form,

$$H = \tfrac{1}{4} W \cot. i$$

and lateral thrusts along the side-rafters are expressed by

$$R = \tfrac{1}{4} W \operatorname{cosec.} i.$$

2°. Proceeding now to the secondary truss, $I N K L$, and considering it isolated from the other trusses, let us suppose it carried by means of the connexions at, I and L, joining it to the primary truss, $I L L_1$. In that case the load pressing on this truss will act through the centre point, K, and will be equal to half the weight lying between, K and I, added to half that between, K and L; or in all to half the weight on the side-rafter, $I L$, and will therefore be expressed by, $\tfrac{1}{2}[\tfrac{1}{2} W] = \tfrac{1}{4} W$.

The distribution of the weight, ($\tfrac{1}{2} W$), supported by the side rafter, is effected by supposing a part, $\tfrac{1}{4} W$, to be transmitted through the centre joint, K; and two parts, each equal to, $\tfrac{1}{8} W$, to be directed through the joints, I and L.

The line of loads for this secondary truss is given in Fig. 46, which is drawn by commencing at a point, X, and setting off in succession, $\overline{X A} = \tfrac{1}{8} W$; $\overline{A C} = \tfrac{1}{4} W$; and $\overline{C Y} = \tfrac{1}{8} W$, and retracing the same line for the two equal reactions, $\overline{Y B}$ and $\overline{B X}$.

To find the reciprocal figure, begin at the nucleus, 1, 5, 6, 10, and draw, from point, A, or junction (1×2), a line 6, parallel to bar, 6; similarly from (4×5) draw a line, 10, parallel to bar, 10. Next construct the reciprocal of nucleus, 3, 4, 7, 8, by drawing from (2×3) a line, 7, parallel to bar, 7; and secondly from (4×5) a line, 8, parallel to bar, 8. It will, then, be evident that the reciprocal of bar, 9, is in the fullest sense determinate, and must coincide with the line, joining the points, F and O.

Analytical Expressions for the Graphic Values.—Draw a line $\overline{CE}$, perpendicularly to, $\overline{AO}$. Similarly, draw $\overline{BG}$ perpendicularly to the same line; $\overline{BH}$ perpendicularly to, $\overline{OF}$; and, $\overline{FD}$, perpendicularly to, $\overline{XY}$.

Since the line, $\overline{CE}$, is perpendicular to $\overline{AO}$; and, $\overline{AC}$, perpendicular to the horizontal line, $\overline{OB}$; it follows that angle

$$A\hat{O}B = i^\circ = A\hat{C}E;$$

wherefore line,

$$\overline{FO} = \overline{CE} = \overline{AC}\, cos.\, i = \tfrac{1}{4}\, W\, cos.\, i.$$

Again,

$$\overline{OB} = \overline{AB}\, cot.\, i = \tfrac{1}{8}\, W\, cot.\, i.$$

But since triangle, OBF, is isosceles;

$$\overline{FB} = \overline{OB} = \tfrac{1}{8}\, W\, cot.\, i.$$

Thirdly,

$$\overline{AO} = \overline{AB}\, cosec.\, i = \tfrac{1}{8}\, W\, cosec.\, i;$$

and

$$\overline{CF} = \overline{OE} = [\overline{AO} - \overline{AE}];$$

or since,

$$\overline{AO} = \tfrac{1}{8}\, W\, cosec.\, i\,;\ \overline{AE} = \overline{AC}\, sin.\, i = \tfrac{1}{4}\, W.\, sin.\, i\,;$$
$$\overline{CF} = \tfrac{1}{8}\, W\, cosec.\, i - \tfrac{1}{4}\, W.\, sin.\, i$$
$$= \tfrac{1}{8}.W\Big[cosec.\, i - 2\, sin.\, i\Big].$$

The above values will be found to correspond with those found by Rankine, (*Applied Mechanics*, § 159) from purely analytical considerations, and the following additional values, deduced from the graphic diagram, similarly agree with the analytical results of the same author.

$$\overline{AO} = \overline{AG} + \overline{GO}$$
$$= \overline{AG} + \overline{BH} = \overline{AB}.\, sin.\, i + \overline{HF}\, cot.\, i.$$
$$= \tfrac{1}{8}\, W.\, sin.\, i + \tfrac{1}{2}\, \overline{FO}\, \frac{1}{tan.\, i}.$$

Similarly,

$$\overline{FC} = \overline{K_1H}$$
$$= \overline{BH} - \overline{BK_1} = \tfrac{1}{2}\, \overline{FO}\, \frac{1}{tan.\, i} - \tfrac{1}{8}\, W\, sin.\, i.$$

Fig. 48.

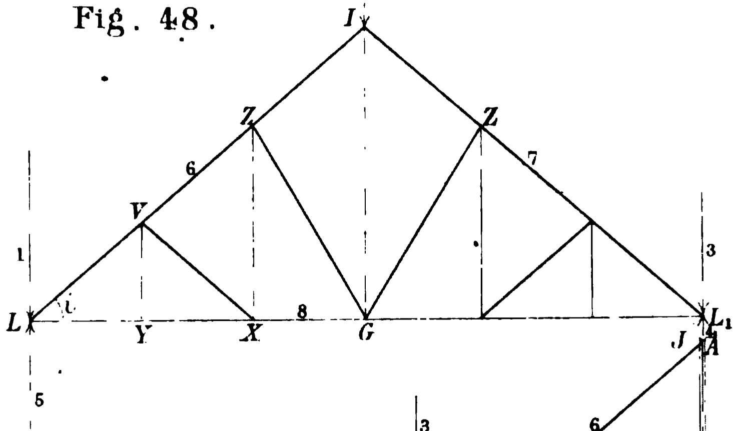


Fig. 49.

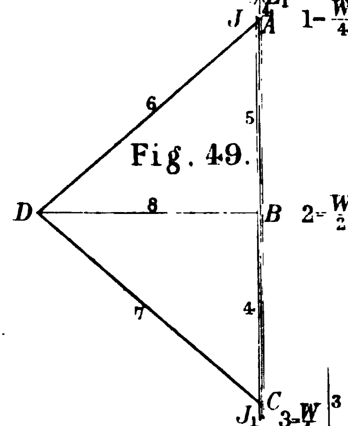


Fig. 50.

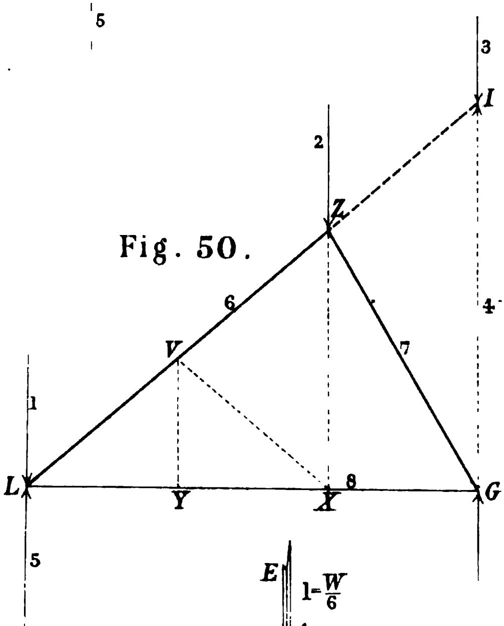


Fig. 51.

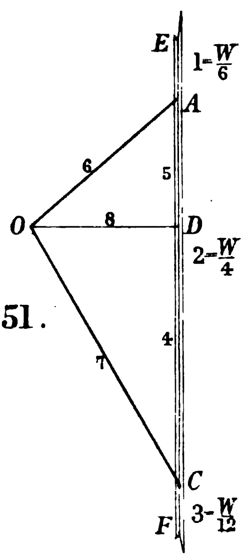


Fig. 52

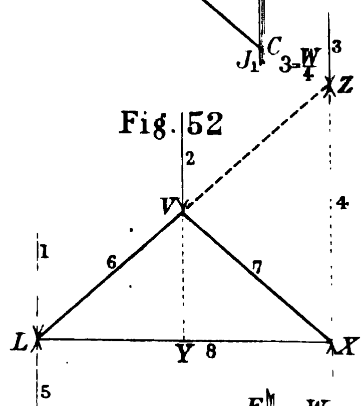
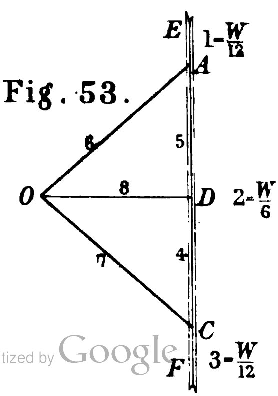


Fig. 53.



If we pass round the quadrilateral figure, $A\,C\,F\,O\,A$, which forms the closed polygon, reciprocal of nucleus, K; commencing with the known direction of load, 2, which is graphically represented by the line, $\overline{A\,C}$, we see that the two stresses, along bars 6 and 7, take place in different senses. At the same time they both constitute thrusts mutually opposed. Hence, the *resultant thrust* at this point, K. will be equal to the difference between the two lines, $\overline{A\,O}$ and $\overline{F\,C}$, which represent the separate thrusts in the reciprocal figure. Consequently if $R =$ the resultant thrust,

$$R = \overline{A\,O} - \overline{F\,C}$$
$$= \overline{A\,E} = \tfrac{1}{4}.\,W\,sin.\,i.$$

The reciprocal figure of each of the two secondary trusses, $K\,M\,L$ and $K\,Q\,I$, would be identical in form with that of the larger truss just treated, the only difference being that the stresses belonging to the truss, $I\,N\,L$, would be in each instance double those of the smaller trusses. Hence, if we suppose Figs. 45, 46, drawn to half the scale, by which they were originally constructed, they will serve to represent the bars and stresses belonging to trusses, $K\,M\,L$ and $K\,Q\,I$.

The stresses, derived from the graphic diagrams of the component trusses, separately taken are *accumulative*;—that is to say, any bar, which forms part of several graphic diagrams, sustains a resultant stress equal to the sum of the lengths of the lines, which reciprocally represent it in the various figures. For example, the bar, $N\,L$, forms part, not only of the primary truss, $L\,I\,L_1$; but also of the secondary truss, $L\,I\,N$. Therefore, the *total stress* along the bar, $N\,L$, will be represented by the sum of the lengths of the lines,

$\overline{O\,H}$ (Fig. 44) and, $\overline{O\,B}$, (Fig. 46).

Similarly, the stress along, $M\,L$, will be equal to the sum of three lines; viz.,

$\overline{O\,H}$ (Fig. 44); $\tfrac{1}{2}\,\overline{O\,B}$, (Fig. 46); and $\tfrac{1}{2}.\ \overline{O\,B}$ (Fig. 46).

The third line, $\tfrac{1}{2}\,\overline{O\,B}$, expresses the partial stress, due to the secondary truss, $K\,M\,L$.

The law of accumulation above explained will be further developed in the next example, in which, for the sake of comparison, we shall construct not only the graphic diagram of each independent truss, but also that of the roof-frame, taken as a whole.

If in Fig. 45, the strut-brace, 9, were omitted, and the reciprocal figure of the roof-frame, deprived of the bar, 9, were constructed, there would be left a gap between points, O and F, Figs. 46, 47. According to the rule given by Rankine, this gap indicates at once where a strut is required and furnishes the stress along the absent bar. (For general diagram see Part II. § 12).

9. ROOF-FRAMES. THIRD EXAMPLE.—Fig. 48, represents another form of roof-truss, with side-rafters, $I\,L$ and $I\,L_1$ and a great tie-rod, $L\,L_1$, Here again as in the last example we shall take the several trusses separately, adding by way of corroboration the graphic diagram of the roof-frame, taken as a whole.

Primary Truss.—Considering first in order the primary truss, $I\,L\,L_1$, the load will be distributed over it in the same manner as in the last example; that is, half the incumbent weight $\left(\frac{W}{2}\right)$ will be concentrated at the ridge-joint, I, and a part, $\frac{W}{4}$, will be applied directly through the supports, L and L_1.

The reciprocal figure of the primary truss is shewn in Fig: 49, which is constructed in the following way:—

Commencing at a point, J, draw a vertical line, $\overline{J\,A}$, equal to force, 1, at, L; or, $\frac{W}{4}$;—from, A, set off in continuation of the same line a second part, $\overline{A\,C}$, equal the force, 2, applied at I; or, $\frac{W}{2}$; and lastly lay off, $\overline{C\,J}$, equal force, 3, at L_1 $\left(\frac{W}{4}\right)$.

Retracing the same line, $\overline{J\,J}$, which is the line of loads, divide it at, B, into two parts, representing the equal reactions at, L and L_1.

Next, between loads, 1 and 2, draw a line, $\overline{A\,D}$, parallel to

the rafter, $I\,L$, and from (2×3) a line, $\overline{C\,D}$, parallel to rafter, $I\,L_1$. These two lines will intersect at, D; from which point draw the line, $\overline{D\,B}$, parallel to the tie-rod, $L\,L_1$. It will be found that the line, $\overline{D\,B}$, will meet the line of loads at a point, B, corresponding to its division in the ratio of the reactions.

The analytical expressions for the various stresses in the bars of the primary truss are found as usual, and can be expressed as follows.

$$\overline{D\,B} = H = \overline{A\,B}\ cot.\ i = \frac{W}{4}\ cot.\ i$$

$$\overline{A\,D} = R = H\ sec.\ i = \frac{W}{4}\ cosec.\ i$$

The stress along the suspension-rod, $I\,G$, arising from the load considered with reference to the primary truss, is seen to be *nil.*

Larger Secondary Truss.—Viewing next in order the larger secondary truss, $G\,L\,Z$, separately taken, we shall suppose it to be held up at, G, by the pull induced along the suspension-link, $I\,G$, which forms part of the primary truss.

With respect to the distribution of the load over this truss, the joint, Z, supports half the weight lying between, Z and I, together with half that between Z and L; or in all half the weight on the side-rafter, $I\,L$, which, expressed in the usual form, will be equal to. $\frac{1}{2}\,[\frac{1}{2}\,W]$; or $\frac{1}{4}\,W$.

Graphic Diagram.—(Fig. 51.) Draw the vertical line of loads, $\overline{E\,F}$, commencing at E, and setting off in succession the part-lines, $\overline{E\,A}$ = force, 1, applied at, L, = weight on $\frac{L\,Z}{2}$, $= \frac{1}{2}\left[\frac{2}{3}\cdot\frac{W}{2}\right] = \frac{W}{6}$;—again, $\overline{A\,C}$ = force, 2, at the joint, Z, which has been shewn to be equal to, $\frac{W}{4}$;—and, $\overline{C\,F}$, = force applied at the point, I, = half weight on $I\,Z = \frac{W}{12}$. Next, from junction (1×2) at A, draw a line, $\overline{A\,O}$, marked, 6, parallel to the side, 6; and from junction (2×3) at C, draw a line, 7, parallel to bar, 7. These two lines will meet in the pole, O, from which draw a line, $\overline{O\,D}$, marked, 8, parallel to bar, 8. This line, $\overline{O\,D}$, will divide the line of loads at, D, into two

parts, which will represent the reactions at, L and I. It will be seen that these reactions are equal to each other and to the common value, $\frac{W}{4}$, graphically represented by either of the equal parts, $\overline{ED}$ and $\overline{DF}$, broken off, on the line of loads.

Moreover, the line, $\overline{OD}$, will divide the line, $\overline{AC}$, corresponding to the load at, Z, into two parts, $\overline{AD}$, and DC, graphically defining the reactions at, L and G, due to the load, $\frac{W}{4}$, at Z.

To find the values of these reactions, $\overline{AD}$ and $\overline{DC}$, in terms of the load at, Z, $\left(\frac{W}{4}\right)$, draw a vertical line, $\overline{ZX}$, meeting, $\overline{LG}$, in, X. Similarly, through, V, draw a line, $\overline{VY}$, meeting, $\overline{LG}$, in, Y.

By similar figures,

$$\frac{AD}{DO} = \frac{ZX}{LX};$$

and,

$$\frac{DC}{DO} = \frac{ZX}{GX};$$

wherefore by division,

$$\frac{AD}{DC} = \frac{GX}{LX} = \frac{IZ}{LZ} = \frac{1}{2};$$

Hence,

$$\overline{AD} = \frac{1}{2} \cdot \overline{DC};$$

that is,

$$\overline{AD} = \frac{1}{3} \cdot \overline{AC} = \frac{W}{12};$$

and,

$$\overline{DC} = \frac{2}{3} \cdot \overline{AC} = \frac{W}{6}.$$

Consequently, the pull induced along the suspension rod, IG, by reason of the weight, $\frac{W}{4}$, lying on the secondary truss, GZL, is equal to the reaction at, G; or by the above analysis to, $\frac{W}{6}$. An equal pull, due to the corresponding secondary

truss on the other side of the centre line, $\overline{IG}$, will cause the expression, $\frac{W}{6}$, to become, $2\left(\frac{W}{6}\right)$; or, $\frac{W}{3}$, which gives the total tension along the link, IG.

The analytical expressions for the other stresses take the forms :—

$$\overline{AO} = \overline{AD}\ cosec.\ i. = \frac{W}{12}.\ cosec.\ i.$$

$$\overline{DO} = \overline{AD}\ cot.\ i. = \frac{W}{12}.\ cot.\ i.$$

$$\overline{OC} = \sqrt{\overline{OD}^2 + \overline{DC}^2}$$

$$= \frac{W}{6}.\sqrt{1 + \frac{cot^2\ i.}{4}}$$

Discussion of the Load W_2 at I.—We have shewn in the isolated treatment of the primary truss, $L\,I\,L_1$, that the total load at, I, is equal to half the weight between I and L added to half that between, I and L_1; or in all to $\frac{W}{2}$. It can be easily shewn that the same load arises from the distribution adopted for the secondary trusses, $G\,Z\,L$ and $G\,Z_1\,L_1$. For, when the secondary trusses are considered apart from the primary truss, the *direct load* applied at I, is for each truss equal to half the weight on $I\,Z$, Fig. 50, or $\frac{W}{12}$; whilst the direct weight concentrated at, Z, is $\frac{W}{4}$. Now, it has been proved that the reaction at, I, due to this particular distribution of the load, is graphically represented by the line, $\overline{FD}$, Fig. 51. But $\overline{FD} = \overline{FC} + \overline{CD} = \frac{W}{12} + \frac{2}{3}.\frac{W}{4} = \frac{W}{4}$. An equal and similar reaction will be induced at, I, due to the load on the corresponding truss, $G\,Z_1\,L_1$, on the other side of the centre-line, $I\,G$.

Consequently the resultant reaction at, I, will be expressed by, $\frac{W}{4} + \frac{W}{4}$; or by $\frac{W}{2}$.

Smaller Secondary Truss.—The smaller secondary truss,

$L\,V\,X$, will be supported at, X, by means of the suspension-link, $\overline{Z\,X}$, belonging to the truss last considered; and at L, by the resistance of one of the pillars.

If a line, $\overline{V\,Y}$, be drawn perpendicularly to the line, $\overline{L\,X}$, Fig. 52, it will be evident that the line, $\overline{L\,X}$, is bisected at, Y; for,

$$\frac{L\,Y}{L\,X} = \frac{L\,V}{L\,Z};$$

But,

$$\overline{L\,V} = \frac{1}{2}\,\overline{L\,Z}$$

Therefore,

$$\overline{L\,Y} = \frac{1}{2}\,\overline{L\,X}$$

With respect to the distribution of the load on this truss, the weight concentrated at, V, will be equal to the sum of the half-loads on, $V\,Z$ and $V\,L$; or in all to half the weight on $L\,Z$, which can be expressed as, $\frac{1}{2}\left[\frac{2}{3}\cdot\frac{W}{2}\right] = \frac{1}{6}\,W$.

Graphic Diagram. (Figs, 52, 53).—Commencing at a point, E, draw the vertical line of loads, making consecutively, $\overline{E\,A}$ = force applied at L, $= \frac{W}{12}$; $\overline{A\,C}$ = force at $V = \frac{W}{6}$; $\overline{C\,F}$ = force at $Z = \frac{W}{12}$.

The reciprocal figure is constructed by the same method as was employed in the case of the larger secondary truss.

The reactions at, L and Z, due to this truss, are represented by the lines, $\overline{E\,D}$ and $\overline{D\,F}$, respectively; and it will be observed by way of comparison that the reaction at Z is equal to, $\overline{F\,D} = \overline{F\,C} + \overline{C\,D} = \frac{W}{12} + \frac{1}{2}\cdot\frac{W}{6} = \frac{W}{6}$; or exactly equal to the incumbent weight which was brought to bear directly on the same joint, considered as due to the weight of the part of the larger secondary truss, between Z and L. In this way, it is made clear to the mind how the different aspects, under which the load is viewed, agree in determining the same resultant load at any particular joint. In one case the distribution

Fig. 54.

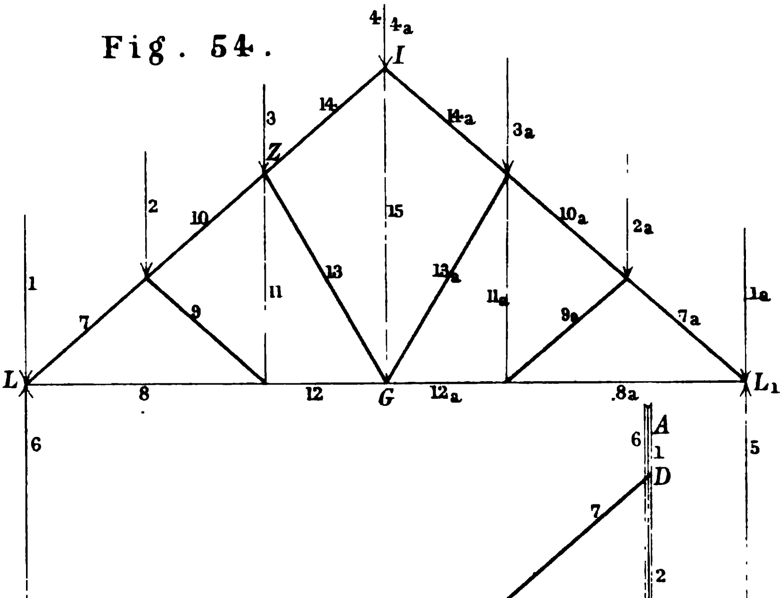
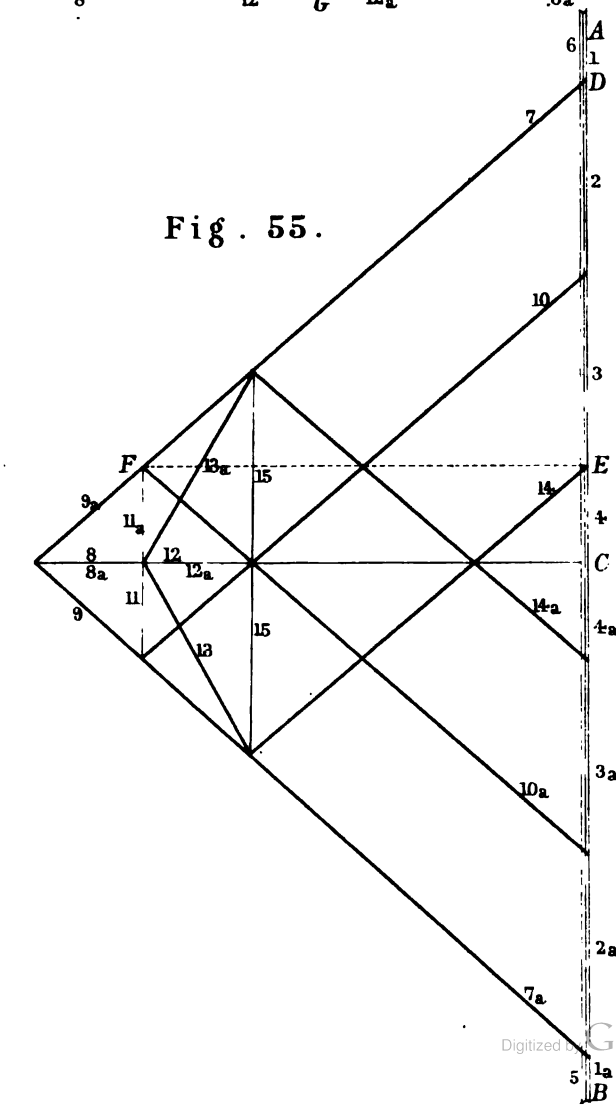

4 4a
I
14 14a
3 3a
Z
2 2a
10 10a
15
13 13a
1 1a
11 11a
9 9a
7 7a
L
L1
8 12 G 12a 8a
6 5

Fig. 55.


A
6 1
D
7
2
10
3
F
13a
15
E
14
4
9a
11a
8 12
8a 12a
C
11
9
14a
4a
15
13
3a
10a
2a
7a
5 1a
B

is direct, and in the other indirect. For instance, going back to Fig. 48, it is quite allowable to suppose either half the weight of the roof concentrated at I, or to imagine only a fractional part *directly* applied at that joint, whilst another part of the load passes immediately through, Z, and then, by means of a double deviation, traverses the rods, $Z\,G$ and $G\,I$, arriving again at, I, by an *indirect* and circuitous path. The second case is analysed at page 55. Returning to the graphic diagram, Fig. 53, it will be seen that the line, $\overline{O\,D}$, divides the line, $\overline{A\,C}$, which represents the direct load at, V, into equal parts, $\overline{A\,D}$ and $\overline{D\,C}$, corresponding to the reactions, at L and X, due to the incumbent weight at, V. Consequently the tension along the link, $Z\,X$, is given by the line of stress, $\overline{D\,C}$, and is, therefore, equal to

$$\frac{1}{2}\left[\frac{W}{6}\right] = \frac{W.}{12.}$$

The analytical expressions for the various stresses along the members of this truss will be :—

$$\overline{A\,O} = \overline{A\,D}.\ cosec.\ i = \frac{W.}{12}\ cosec.\ i$$

$$\overline{D\,O} = \overline{A\,D}\ cot.\ i = \frac{W}{12}\ cot.\ i$$

The resultant stresses along the bars are made up of the sums of the separate stresses derived from the graphic diagrams of the independent trusses, of which the particular bars form parts. For instance, the bar, $G\,X$, is common to the trusses, $L\,I\,L_1$ and $G\,Z\,L$;—consequently the resultant stress along it will be represented by,

$$\overline{B\,D}\,(\text{Fig. } 49) + \overline{O\,D}.\ (\text{Fig } 51)\,;$$

or by

$$\frac{W}{4}\ cot.\ i + \frac{W}{12}\ cot.\ i = \frac{W}{4}\ cot.\ i\left[1 + \frac{1}{3}\right]$$
$$= \frac{W}{3}\ cot.\ i$$

On the same supposition the resultant stresses along the other bars can be found by a similar process of summation.

For example the sum of the stresses along the bar, $L\,X$, must be derived from the reciprocal figures of the trusses, $L\,I\,L_1$, $G\,Z\,L$, and $L\,V\,X$, of all of which it forms a part. Hence, the resultant stress along this bar will be defined by the graphic sum,

$$\overline{B\,D}\ (\text{Fig. } 49) + \overline{O\,D}\ (\text{Fig. } 51) + \overline{O\,D}\ (\text{Fig. } 53);$$

or by,

$$\frac{W}{4}.\,cot.\,i + \frac{W}{12}.\,cot.\,i + \frac{W}{12}.\,cot.\,i. =$$
$$= \frac{W}{12} cot.\,i.\ [3+1+1]$$
$$= \frac{5}{12}\,W.\ cot.\,i.$$

Graphic Diagram of the Complete Roof Frame.—Let us now construct the general reciprocal figure of the same roof-frame (Fig. 54) viewed as a whole, in order to find the resultant stresses by a more direct and immediate process. The stresses so found must agree with the values determined for them by the process of disintegration and summation developed in the preceding pages.

The half weight $\left(\frac{W}{2}\right)$ on each rafter will be distributed according to the following law :—

At the ridge-joint, I, there will be concentrated a load equal to half the weight lying between force-lines, 4 and 3. Similarly, for the other rafter, a load equal to the half-weight between lines, $4\,a$ and $3\,a$, will be brought to bear upon the ridge-joint. In all, therefore, the ridge-joint, I, will sustain a load equal to the weight lying between the force-lines, 4 and 3.

Again, at joint, 3, there will be concentrated a weight equal to half the load between lines, 3 and 4 ; 2 and 3 ; or, in other terms, a weight equal to that between any two of the force-lines. The load on joint, 2, is the same as that on joint, 3 ; and at 1 or L, the load is equivalent to half the weight between force-lines, 1 and 2 ; or to half that at the other joints of the rafter. The above distribution of the load

is marked on the line of loads, $\overline{AB}$, Fig. 55, in which the divisions 1, and 1 a, are made equal to half of each of the other divisions on the same line. The load on the ridge-joint is seen to be divided at, C, into two parts, 4 and 4 a, derived separately from the loads on the side-rafters.

The construction of the reciprocal figure offers no difficulty. Commencing at the nucleus, (1, 6, 7, 8), draw between junction (1×2), a line, 7, parallel to bar, 7; and from junction (5×6), a line, 8, parallel to bar, 8. These two lines will intersect and form the closed figure, (1, 7, 8, 6) reciprocal of the nucleus similarly described.

Proceeding next to nucleus, (2, 7, 9, 10), draw from (2×3) a line, 10, parallel to bar, 10; and from (7×8) a line, 9, parallel to bar, 9.

The rest of the figure is constructed in precisely the same way, and can, therefore, be completed by the reader himself.

On making a comparison of the different figures, and taking the necessary measurements by means of a pair of compasses, it will be found that the lines of the resultant reciprocal figure are the linear sums, of the lines, correlative of them in the component reciprocal figures of the trusses separately taken. For example,

Line, 8, Fig. 55, $= \overline{BD}$, Fig. 49, $+ \overline{OD}$, Fig. 51, $+ \overline{OD}$, Fig. 53

$$= \overline{DC}.\ cot.\ i = \frac{5}{12}\, W\ cot.\ i \quad \text{(See p. 60.)}$$

Again, since bar, 13, forms part of only one of the separate trusses; viz.. GZL, its reciprocal, B, Fig. 55, should be equal to its reciprocal, 7, Fig. 51.

For the same reason, line, 9, Fig. 55, is equal to line, 7, Fig. 53.

To take a fourth example, the line, 12, will be equal to the graphic sum,

$$\overline{BD}, \text{ Fig. 49, } + \overline{OD}, \text{ Fig. 51.}$$

or to,

$$\frac{W}{3} \cdot cot.\ i. \quad \text{(See p. 59.)}$$

Now, by Fig. 55.

$$\begin{aligned} \text{Line, 12,} &= \overline{FE} \\ &= \overline{DE}\,cot.\,i \\ &= \frac{W}{3}\,cot.\,i\,; \end{aligned}$$

so that we may conclude that the stresses, found by the two methods, coincide and are confirmatory one of the other.

10. Straight Link Suspension.—In Fig. 56, is shewn a truss, which often forms part of Suspension Bridges. As can be seen in the figure, part of the platform is hung from the head of the neighbouring tower by means of straight links.

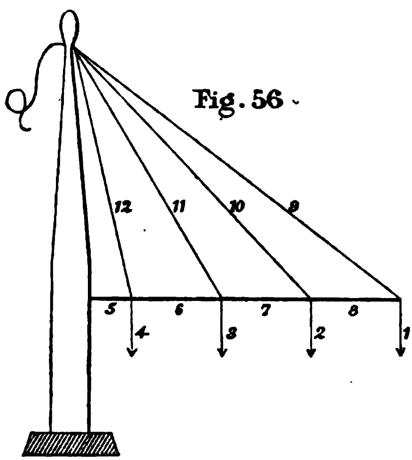


Fig. 56.

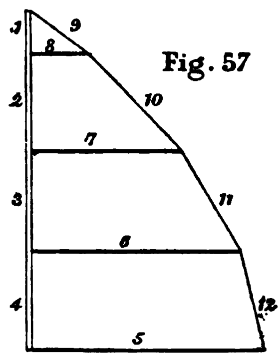


Fig. 57

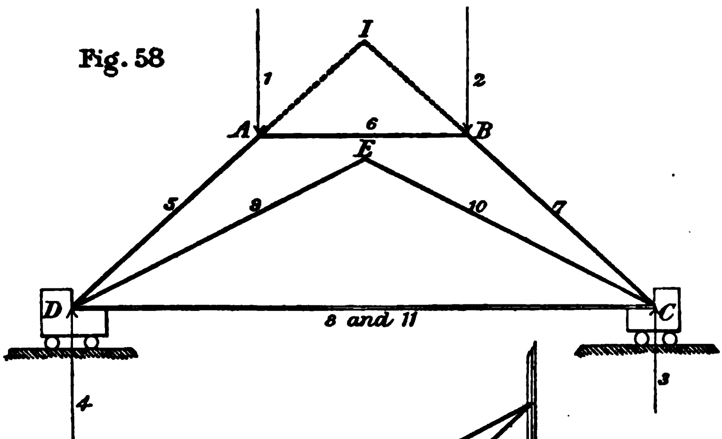
Fig. 57, gives the stresses, which are seen to be thrusts along the sections of the platform, and pulls along the suspending links. Link, 9, suffers the least, and link, 10, the greatest tension. Section, 8, bears the least, and section, 5, the greatest thrust. As regards the tensions, it will be seen that for the same load they become greatest in those rods which are most inclined to the vertical, the diminution in link, 9, being due to the lightness of the load on the end section, 8.

11. Compound Bridge Truss.—Fig. 58, represents a kind of frame-work sometimes used in timber structures. It consists

of two distinct trusses, one triangular the other trapezoidal, having a common tie-bar, *D C*.

A truss of this kind must be taken to pieces, and treated by the method of disintegration and summation already explained for roof-frames.

Fig. 58



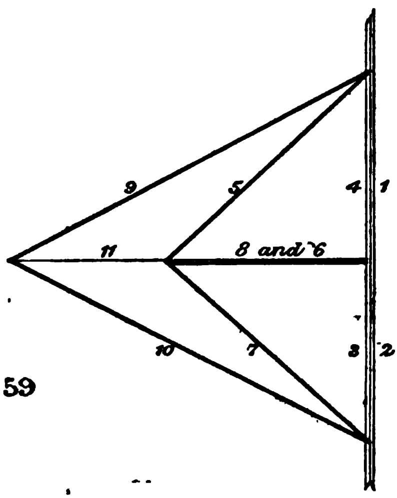


Fig. 59

Taking first in order the trapezoidal truss, we may suppose two equal loads applied at the joints, *A* and *B*, whilst the other two joints of the frame, *D* and *C*, rest on the piers.

Drawing the reciprocal, as given in Fig. 59, we find that the stresses along bars, 6 and 8, are equal in magnitude but opposite in sense;—8, being in tension, and, 6, in compression.

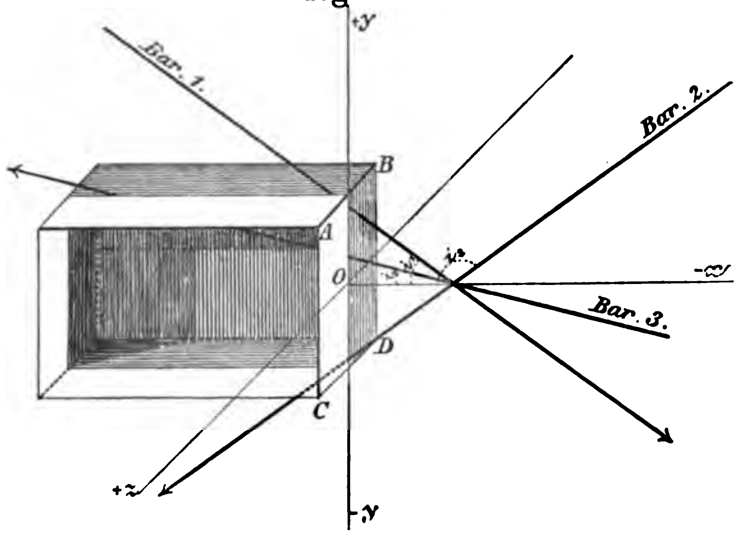
It will be observed that the stress along bar, DC, is tensional and equal to the sum of the pulls due to the two trusses; or to (8 + 11), Fig. 59.

In the case of the trapezoidal frame, the usual criterion, expressed by the formula,

$$S = 2p - 3,$$

is not satisfied; but the reciprocal figure is none the less obtainable, because in reality a trapezoidal frame is equivalent

Fig. 60



to a triangular frame, formed by omitting the bar, 6, and producing the lines, DA and CB, to meet in a point, I. The whole load would then be supposed to be applied at, I.

Any number of triangular or trapezoidal trusses, compounded together in this way, could be treated by a similar method.

12. Graphic Interpretation of the Method of Sections.—In this article we shall explain the method of sections, first analytically, and secondly by the aid of graphic principles; so that the two methods being put side by side may be intelligently compared.

The frame-work and the force-lines will be taken in one plane.

Selecting in this way, the lines of resistance of three bars, Fig. 60, and their correlative force-lines, situate in one plane, which in the present instance forms the plane of the paper; let us suppose this plane to be intersected by another lying at right angles to itself. Take the line of intersection of these two planes as axis of, y, and a point, O, in the axis of y, so determined, as the origin of co-ordinates. Let the axis of, x, be perpendicular to the axis of y, and in the plane of the bars; and the axis of, z, perpendicular to the plane, $x\,y$, and situate in the plane of section, $A\,B\,C\,D$.

The external forces applied on either side of the plane, $A\,B\,C\,D$, say to the left, can be resolved into rectangular components acting along the axes of x and y respectively. Let the resultant of these forces or the algebraic sum of the separate forces resolved along the axis of x be represented by

$$\Sigma.\,F.\,cos.\,a = F_x. \qquad (1).$$

Similarly, let the resultant force along the axis of y, be represented by

$$\Sigma.\,F.\,sin.\,a = F_y. \qquad (2).$$

Let, L, equal the perpendicular distance from the origin, O, of any force-line, F; then the resultant moment of the system of forces, relatively to the axis of Z, will be expressed by

$$\Sigma.\,F\,L = M. \qquad (3).$$

The usual static conditions of equilibrium require that the bars, which are cut by the plane of section, $A\,B\,C\,D$, should exert resistances capable of balancing the two forces, F_x and F_y, and the resultant couple, M.

The equations, marked, (1), (2), and (3) will help to determine three unknown resistances; but, if more than three bars are cut by the plane of section, the problem becomes indeterminate.

Let us, therefore, suppose that the plane, $A\,B\,C\,D$, Fig. 60,

cuts only three bars, marked, 1, 2, 3;—let, x, measured to the left of the plane of section, be considered positive;—let, y, upwards be positive; let angles measured from $+x$ towards $+y$, be made positive; and let the centre-lines of the three bars, cut by the plane of section, form angles, i_1, i_2, and i_3 with the axis of x. Finally, let the perpendicular distances of the three bars from the origin, O, be represented by, d_1, d_2, and d_3; distances to the right of, Ox, being made positive and those to the left negative; and let, R_1, R_2, and R_3, stand for the total stresses along the bars, pulls being positive and thrusts negative.

The first condition of equilibrium demands that the resultant of the external forces resolved along the axis of, x, be equal and opposed to the resultant of the resistances resolved along the same axis. This condition is expressed by the equation

$$F_x = R_1 \cos. i_1 + R_2 \cos. i_2 + R_3 \cos. i_3.$$

The second condition is similar to the first, the forces and resistances being resolved along the axis of, y, and can be put in the form,

$$F_y = R_1 \sin. i_1 + R_2 \sin. i_2 + R_3 \sin. i_3.$$

The third condition requires that the sum of the moments of the forces around the axis of, z, be equal in magnitude but opposite in sense to the sum of the moments of the resistances around the same axis;—which, symbolically expressed, takes the equational form

$$-M = R_1 d_1 + R_2 d_2 + R_3 d_3.$$

From these three equations of equilibrium the values of the unknown resistances can be found by a process of elimination. In using this method it is very necessary to distinguish the directions of the forces and their position and inclination, relatively to the same base-line, xx, by appropriate signs. The same problem can be treated by the help of graphic methods, and in fact the general case of this problem has already been solved in Part I. Chap. I. § 7. Here it will be only necessary to

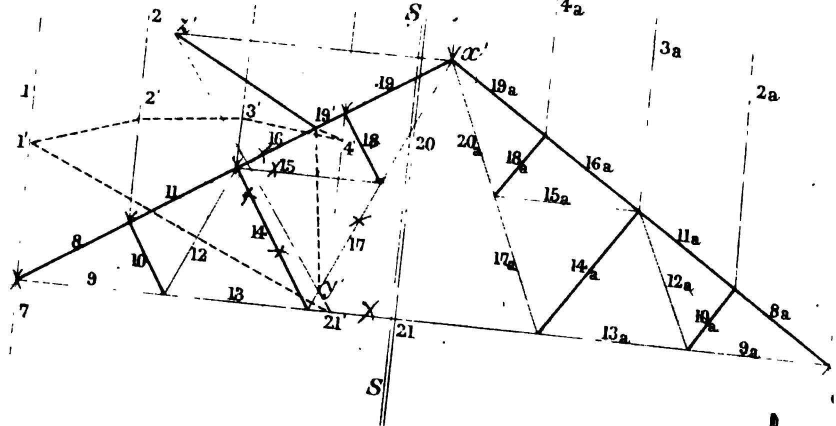


Fig. 61.

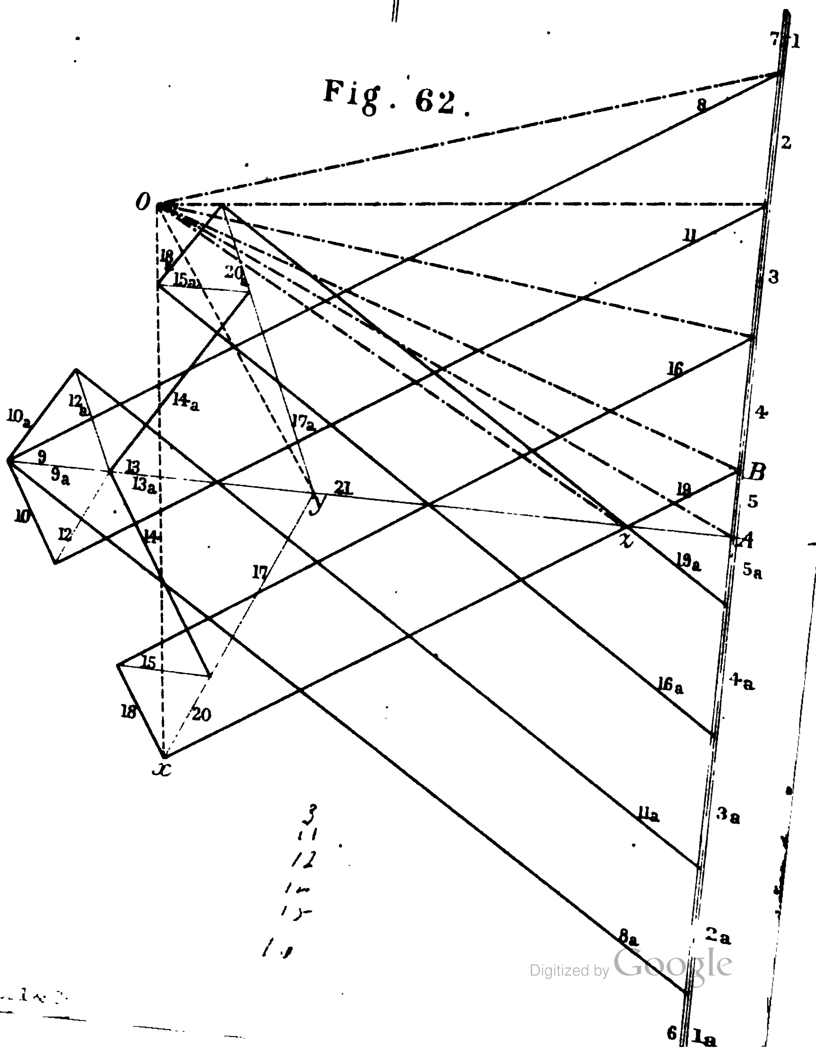


Fig. 62.

adapt the general method given in Part I. to a particular case, in which all the applied forces are taken as parallel. Generally viewed the problem takes the shape of finding the stresses induced by a given system of forces along three lines given in direction. The actual and concrete case at present before us will be best explained by taking a familiar example.

Let it be required to find the general reciprocal figure of the roof frame, treated separately truss by truss in Part II. § 8.

If we commence to build up the general reciprocal figure (See Figs. 61, 62), according to the usual method, a difficulty will arise and arrest our progress. It is easy enough to find the closed polygons, reciprocal of nuclei, (1, 7, 8, 9); (2, 8, 10, 11); and (9, 10, 12, 13); but here we must pause; for, whether we pass next to nucleus, (3, 11, 12, 14, 15, 16); or to, (13, 14, 17, 21), we meet with three lines, the reciprocals of which are unknown.

To escape from the difficulty, it becomes necessary to apply either the analytical method of sections, or an equivalent graphic process.

Consequently, take a section, *S S*, Fig. 61, intersecting the three bars, 19, 20, 21. The problem resolves itself into that of determining the stresses induced along the bars, 19, 20, and 21, by the series of vertical forces, 1, 2, 3, 4, and 7, acting to the left of the sectional plane, *S S*.

Applying the method detailed in Part I, and already referred to, proceed as follows:—

1°. Pitch any pole, *O*, Fig. 62, and draw the polar lines, $\overline{OA}$, O_{12}, O_{23}, O_{34}, and $\overline{OB}$, the points, *A* and *B*, being the origin and extremity of the line of forces, 7, 1, 2, 3, 4 (the force, 7, being a reaction).

2°. Construct the corresponding polar polygon, 21′, 1′, 2′, 3′, 4′, 19′, Fig. 61.

3°. From, *B*, Fig. 62, draw a line, 19, parallel to bar, 19; and from, *A*, a line parallel to bar, 21;—let these lines intersect at *z*. Join *Oz*.

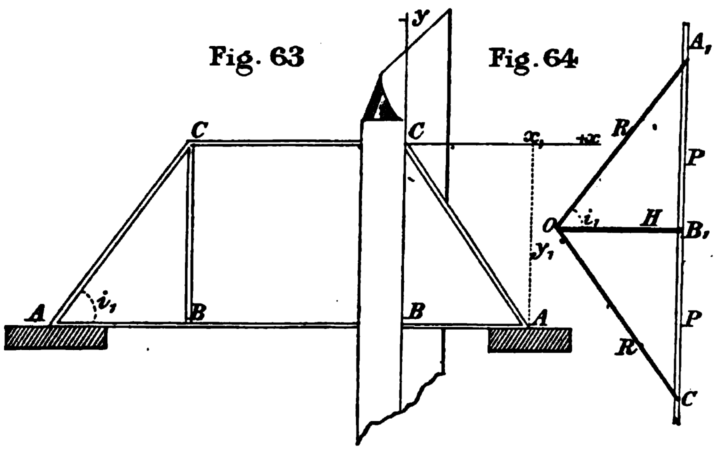
4°. From the ridge-joint, *x′*, at which lines, 19 and 20,

intersect, draw a line, $x' z'$, parallel to bar, 21; and from point, 19′, a line, 19′, z', parallel to polar line, Oz. Let these two lines intersect at, z'.

5°. Join, z', and 21′, by a line intersecting bar, 20, or its prolongation in, y'. Join also 19′, and y'.

Lastly, from pole, O, draw a line, Ox, parallel to, 19′, y' meeting the line, Bz, produced in, x; and a line, Oy, parallel to 21′, y', intersecting, Az, produced in, y. Join, x and y.

Fig. 63 Fig. 64



The required stresses along the bars, 19, 20, and 21, will be graphically represented by the three lines Bx, xy, and Ay, which are numbered accordingly.

The construction of the rest of the figure can be completed, as shewn, by following the usual method.

It will be found on comparison that the stresses found by the method of disintegration and summation, given in Part II. § 8, pp. 48–54, agree with the same values determined by the more direct and immediate process just developed.

For example ;—

The Resultant Stress, 19, Fig. 62, = Sum of stresses, R, Fig. 44; 7, Fig. 46; and $\frac{1}{2}$ of 7, Fig 46.

The Resultant Stress, 20, Fig. 62, = Sum of stresses, 8, Fig. 46; and $\frac{1}{2}$ of 8, Fig. 46.

The Resultant Stress, 21, Fig. 62 = Stress, *H*, Fig. 44.
Stress, 17, Fig. 62, = Stress, 8, Fig. 46.
Stress, 14, Fig. 62, = Stress, 9, Fig. 46.
Stress, 10, Fig. 62, = $\frac{1}{2}$ Stress 9 or 14.

The other stresses can be compared in a similar manner, bearing in mind that the stresses corresponding to the smaller secondary trusses are half those belonging to the larger ones. This has been explained in the article already referred to, where the process of disintegration was applied to this example of roof-frame.

If the analytical method were used, and the origin of co-ordinates were taken at the point, where the line, 21, pierces the plane of section, we should have

$F.\,x = \Sigma.\,F.\,cos.\,a = O.$
$F.\,y = \Sigma.\,F.\,sin.\,a =$ algebraic sum of vertical forces.

As regards the moment about the axis of *Z*, it would take the form,

$$-M = (7-1)\,\delta_7 - 2.\,\delta_2 - 3\,\delta_3 - 4\,\delta_4\,;$$

in which expression any term, 2. δ_2, means the force, 2, multiplied by its perpendicular distance, δ_2, from the origin of co-ordinates. Consequently, the three equations necessary to determine the resistances, R_{19}, R_{20}, and R_{21}, can be put into the following forms, the symbols, δ_{19}, δ_{20}, and δ_{21}, having the same meaning as before defined.

$$F\,x = O = R_{19}\,cos.\,i_{19} + R_{20}\,cos.\,i_{20} + R_{21}.$$
$$F\,y = V = R_{19}\,sin.\,i_{19} + R_{20}\,sin.\,i_{20} + O.$$
$$-M = (7-1)\,\delta_7 - 2\,\delta_2 - 3\,\delta_3 - 4\,\delta_4 = R_{19}\,\delta_{19} + R_{20}\,\delta_{20}.$$

The term, *V*, introduced into the second equation, must be interpreted to mean *the algebraic sum of the vertical forces*, 7, 1, 2, 3 and 4.

13. METHOD OF SECTIONS. SECOND EXAMPLE.—The following problem is one of the simplest examples, which could be chosen, to illustrate the two methods.

Fig. 63, represents a truss, consisting of three struts, $A\,C$, $C\,C$, $C\,A$, a tie-beam, $A\,A$, and two suspension rods, $C\,B$ and $C\,B$. (Rankine, *Applied Mechanics*, p. 152).

The reciprocal figure is easily constructed in this case, and does not require the use of the auxiliary method employed in Ex. I. The stresses along bars, $C\,C$ and $A\,A$, are equal in

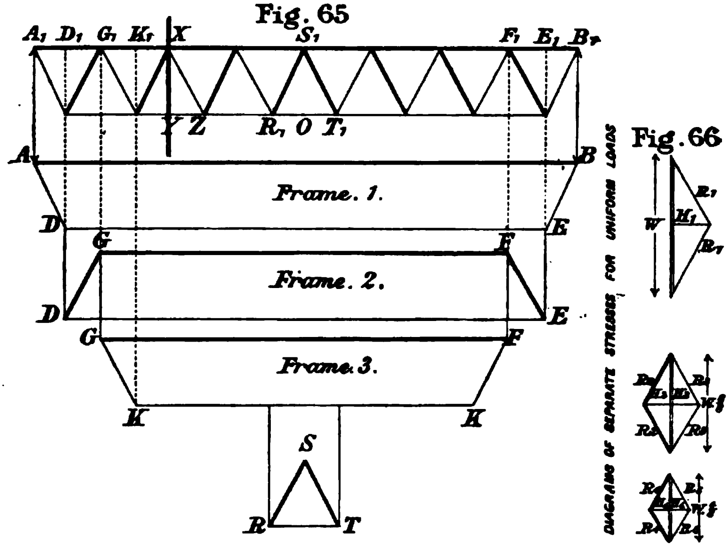


Fig. 65

Fig. 66

magnitude but opposite in sense, having a common graphic value, $\bar{O}\,B_1$, Fig. 64. Leaving out of consideration the portions of the loads bearing directly on the props at, A and A, let P and P denote equal vertical loads applied at, C and C; whilst, $-P$ and $-P$, are the equal reactions at, A and A, due to these parts of the load.

Let, i_1, denote the equal and opposite inclinations of the rafters to the horizontal tie-beam, $A\,A$.

Let a sectional plane pass through the joint, C, Fig. 63, and consider this plane to intersect the bars, $A\,A$ and $C\,A$. Take the centre, C, as the origin of co-ordinates, x being positive towards the right and, y, positive downwards. Let x_1, y_1, be the co-ordinates of the prop, A, to the right of the plane of section, and imagine this plane to be in equilibrium under

the action of the forces applied between it and the nearest prop; so that we have the following relations.

$$F x = 0 \;;\; F y = -P.$$
$$M = -P x_1.$$
or
$$-M = P x_1.$$

Consequently,

$$F x = 0 = R_1 \cos. i_1 + R_2 \cos. i_2,$$
$$F y = -P = R_1 \sin. i_1 + R_2 \sin. i_2,$$
$$-M = P x_1 = R_1 \delta_1 + R_2 \delta_2;$$

in which equations, R_1 and R_2, mean the resistances along $C A$ and $A A$, respectively.

But,

$$i_2 = 0 \;;\; \delta_1 = 0 \;;\; \delta_2 = y_1.$$

therefore

$$F x = 0 = R_1 \cos. i_1 + R_2 \quad (1).$$
$$F y = -P = R_1 \sin. i_1 \quad (2).$$
$$-M = P x_1 = R_2 y_1; \quad (3).$$

whence from (3) and (1)

$$R_2 = P \frac{x_1}{y_1} = P \cot. i_1$$

and,

$$R_1 = \frac{-R_2}{\cos. i_1} = -P \operatorname{cosec.} i_1.$$

In this instance the two methods are not comparable as regards simplicity and dispatch, the algebraic method becoming long and tedious; whilst the graphic process is direct and self-evident. [See Fig. 64.]

14. HALF LATTICE GIRDER.—Fig. 65, represents what is sometimes called a half-lattice, and sometimes a Warren Girder. The type is sufficiently well known and needs no description.

The stresses induced in the members of this frame-work by reason of a series of equal loads, or a load unequally distributed,

can be found by employing any one of three methods; viz., the method of disintegration; that of analytic statics; or that of graphic statics.

If the first method be chosen, the girder-frame must be separated into its component trusses, some of which are given in, Fig. 65.

The distribution of the load is effected by supposing $\frac{1}{8}$th of the total load to be concentrated at the joints of the lower boom of the girder.

Frame, I., is a trapezoidal truss; so is Frame, II., and the remaining trusses down to the central one, $R_1\ S_1\ T_1$, which is triangular.

Considering that the loads, concentrated at the lower joints, result from a uniform distribution of, $\frac{W}{8}$, at each joint, it will be seen that Frame, I., is brought under the whole load; Frame, II., under $\frac{2}{8}$ths less than Frame I.; or under $\frac{6}{8}\ W$;—Frame, III. supports the same load as Frame, II.; viz. $\frac{6}{8}\ W$, since as yet only two of the unit-loads at the lower joints have been cut off from each end of the girder. In like manner, Frames IV. and V. are each subject to a load equal to, $\frac{4}{8}\ W$; and Frames, VI. and VII., to a load expressed by $\frac{2}{8}\ W$. The central triangular truss, $R_1\ S_1\ T_1$, supports no weight; since no separate load is supposed to be applied at the upper joint, S_1.

The reciprocal diagrams of stress for Frames, I.; II. and III.; IV. and V., are given on the right of the girder-outline, being in each instance similar to each other. It will be further apparent that the diagonal stresses, due to Frames, II. and III., are equal in intensity, but opposite in sense. This means that, while the diagonal stresses in one Frame are tensional and situated on the right of the vertical line of loads, in the other (II.) they are compressive and situated on the left of the same line (see Reciprocal Figure for load,

$\frac{6}{8} \cdot W$). The same remarks apply in the case of Frames, IV. and V.

The reciprocal figures shew that the stresses along the horizontal members of the girder go on increasing from the ends towards the central point, O; since the middle section, $R_1 T_1$, enters as a component part into all the trapezoidal trusses preceding the triangular truss, to which it especially belongs; and, therefore the stress along it would be equal to the sum of the stresses given by the reciprocals of the several independent frames.

On the other hand, the diagonal stresses diminish towards the centre of the girder; since any chosen diagonal will be seen to form part of only one separate truss, and, moreover, the inclined stresses are greater in these trapezoidal frames, which are more distant from the centre of the bridge or span.

Let, H_1 and R_1, represent the horizontal and inclined stresses in Frame, I.; and, H_2 and R_2, the same values for Frame, II.; —then, if the angle at which the diagonals are inclined to the horizontal be called, i, we have (Fig. 66),

$$H_1 = \frac{W}{2} \cot. i; \; R = \frac{W}{2} \operatorname{cosec.} i$$

and,

$$H_2 = \frac{6}{16} \cdot W \cot. i; \; R_2 = \frac{6}{16} \cdot W \operatorname{cosec.} i.$$

On examining the given reciprocal figures, it will be seen that the horizontal stresses, H, lie alternately to the right and left of the line of loads. To embrace this circumstance, and to distinguish the inclined stresses into pulls and thrusts, some convention would have to be established with respect to the sines and cosines of the angle, i, which, however, it is unnecessary to go into here.

The stresses in the third frame would take the same form as those in Frame, II.; viz.,

$$H_3 = \frac{6}{16} \cdot W \cot. i; \; R_3 = \frac{6}{16} \cdot W \operatorname{cosec.} i.$$

The remaining stresses would be similarly expressed, those derived from the central truss, $R_1 S_1 T_1$, taking the zero forms,

$$H_8 = \frac{0}{16} W \cot. i\,; \quad R_8 = \frac{0}{16} W \operatorname{cosec.} i.$$

Summing all the horizontal stresses, in order to find the resultant stress along the bar, $R_1 T_1$, we obtain,

$$H_1 + H_2 + H_3 + H_4 + H_5 + H_6 + H_7 + H_8 = W \cot. i \left[\tfrac{8}{16} + \tfrac{6}{16} + \tfrac{6}{16} + \tfrac{4}{16} + \tfrac{4}{16} + \tfrac{2}{16} + \tfrac{2}{16} + \tfrac{0}{16}\right]$$
$$= \frac{32}{16} \cdot W \cot. i = 2. W \cot. i$$

The diagonal stresses diminish from the ends towards the centre of the girder, bearing a maximum value, $\frac{W}{2} \operatorname{cosec.} i$, in the extreme diagonals, and a minimum, $\frac{0\,W}{16} \operatorname{cosec.} i$, or zero, in the central diagonals, $R_1 S_1$ and $S_1 T_1$.

It is, however, much simpler to employ the direct graphic process, detailed for this class of girder in Part I. § 7.

The value of the bending moment, as well as the shearing force at any section, can be indirectly deduced from the graphic diagram, as we shall have occasion to shew in the treatment of this girder by the ordinary analytic method.

The usual algebraic process for finding the stresses in a Warren girder essentially consists in applying the method of sections to each division of the span, supposed to be divided into convenient parts.

The load at each joint being known, the reactions at the supports, A and B, are determined by help of the principle of the equilibrium of parallel forces acting in one plane, which has been explained at the beginning of Part II. Let, P_A and P_B, denote those reactions and, downward loads being made negative, let $-P$ denote the load at any joint of the girder.

Let it be required to find the stress along any one of the diagonals, XZ. Along the upper boom immediately to the right of, X, choose a point and draw a perpendicular line, $\overline{XY}$,

representing the trace of the vertical plane of section. Take the point, X, as the origin of co-ordinates.

The plane of section, $\overline{XY}$, is in equilibrium under the action of the forces applied to the girder between the origin of co-ordinates and the end of the girder, A, both inclusive.

Let, x, be the abscissa of any joint to the left of the origin, x_1 being that of the point of support, A. Let, y, be positive upwards; so that the ordinate of the lower boom will be, $y = -h$; h, being the distance between the upper and lower booms.

Let, i, be the inclination of the diagonal, XZ, angles to the right of the axis of x, being positive.

Let the symbol, $-\Sigma_X^A P$, denote the sum of the loads acting downwards at the joints between X and A, both inclusive.

Adopting the terminology introduced in Part II. § 12, we have, since all the forces are vertical,

$$F_x = O\,;\; F_y = P_A - \Sigma_X^A P.$$
$$M = P_A x_1 - \Sigma_X^A Px.$$

The *shearing force*, Fy, will be positive or negative, upward or downward, according as P_A is greater or less than, $\Sigma_X^A P$.

The *bending moment*, M, is always positive and right-handed; for, it will be observed, the expression for this moment is necessarily positive at the point, A, and as the plane of section travels from, A, towards the right over short intervals, equal in length to, dx; we may suppose that it passes and leaves to its left a new downward force, $-P$, for every distance, dx, passed over. Consequently, at each step in the progress of the plane of section as it moves along the girder, a negative term, $P\,dx$, will be gathered into the general expression for M. But, at the same time, it will receive the addition of a positive term, $P_A\,dx$, and it could easily be shewn that the term, $P_A\,dx$ is necessarily greater than, $P\,dx$. Consequently the expression for the moment remains positive.

Let, R_1, denote the stress or resistance along the upper boom at, X; R_2, that along the lower boom at, Y; and, R_3, that

along the diagonal, XZ. Then the equations of Part II. § 12, allow us to state the following relations :

$$(a).\quad F_x = 0 = R_1 + R_2 + R_3 \cos. i\,;$$
$$(b).\quad F_y = P_y - \Sigma_X^A P = R_1 \times \sin. o + R_2 \times \sin. o + R_3 \sin. i\,;$$
$$(c). -M = -[P_A x_1 - \Sigma_X^A Px] = R_1 \times o + R_2 \times -h + R_3 \times o\,;$$

Fig. 67.

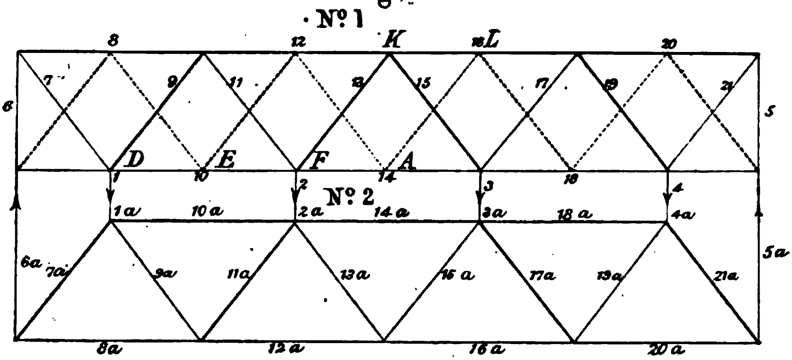
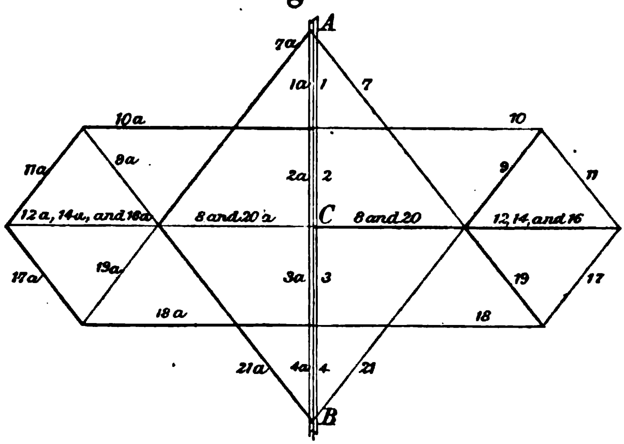


Fig. 68.



which, being simplified, take the forms,

$$(a_1)\quad F_x = 0 = R_1 + R_2 + R_3 \cos. i\,;$$
$$(b_1)\quad F_y = P_A - \Sigma_X^A P = R_3 \sin. i$$
$$(c_1)\quad M = P_A x_1 - \Sigma_X^A Px = R_2 . h$$

By a process of elimination the three resistances can be found by aid of these three equations. Equation, (b_1), forms a connecting link between the graphic and algebraic methods; for, it will be seen that at any sectional plane, if, Fy, represent the local shearing force, and R, the diagonal stress, we have the relation,

$$Fy = R \sin. i;$$

Now the reciprocal figure, characteristic of the graphic process, furnishes the *direct stress*, R, along any diagonal; and, therefore, to find the shearing force in the immediate vicinity of this inclined stress, it is only necessary to multiply its value, as given on the graphic diagram, by the sine of its inclination to the horizontal.

Similarly, we see by equation, (c_1), that if, R_2, be the horizontal stress along any division of the lower boom of a lattice girder, and, M, the bending moment in its immediate vicinity, then

$$M = R_2 . h;$$

so that, to find the actual value of, M, in lattice girders, it is only necessary to multiply the amount of the horizontal stress, R_2, given by the graphic diagram, into the depth, h, of the girder at the division in question.

From the preceding remarks we can deduce two general principles:—

1°. The shearing stress will be least where the diagonal stress reaches a minimum in the reciprocal figure, representing, by the graphic method, the direct stresses along all the members of the girder.

2°. The bending moment is greatest, where the horizontal stresses reach a maximum in the reciprocal figure of stress.

15. THE LATTICE GIRDER.—The lattice girder (Fig. 67) is simply the combination of two half-lattice girders, one being the inverted form of the other. The union of forms, No. 1 and No. 2, Fig. 67, will give rise to a lattice form. Moreover, it will be found that the stresses due to these two forms, separately taken, are the same for the same load, except that

the senses of the inclined stresses are reversed in the two cases.

Now, if the whole load, taken first as distributed over the half-lattice girder, No. 1, be afterwards supposed spread over the two forms; so that half the load rests on the lower joints of No. 1, and the other half on the upper joints of No. 2, we shall have half the load distributed over the upper and half over the lower joints of the combined lattice form.

The construction necessary to find the reciprocal figure of the combined form or lattice girder consists in graphically adding together the stresses induced in the half lattice forms, granting that each separate form supports independently half the load.

Graphic Diagram of a Lattice Girder.—Draw any vertical line, $\overline{AB}$, Fig. 68, representing the half load, to the right of which construct the figure, reciprocal of form, No. 1;—similarly to the left of the same line, draw the reciprocal of form No. 2.

Then, the resultant stress along any bar of the compound girder will equal the sum of the stresses, brought to bear upon it by the loads upon the separate half-lattice forms. For example, the resultant stress along a part, DE, will be equal to the sum,

$$10 + 8_a;$$

which, indeed, is evident from the fact that the division, DE, is a part common to the triangles, (9, 10, 11), No. 1, and (7_a 8_a 9_a), No. 2. Moreover, both the component stresses are tensional; and, therefore, the resultant stress will be also tensional. Similarly, the resultant stress along the part, FA, will be equal to the sum,

$$14 + 12_a,$$

which are both tensions. Along the part, KL, the total stress will be expressed by the sum,

$$16 + 14_a,$$

which is thus composed of two compressive stresses.

The resultant stress along any other division of the girder can be determined by examining in a similar way the other lines of the combined reciprocal figure.

EXAMPLES.

1. Divide the roof-structure, *Swansea Station*, Fig. 165, Pl. I. into its component trusses, and find the nature and amount of the reaction along bar, 14, considered as part of the smaller secondary truss, *A, B, C.*

Reaction = $-1\frac{1}{4}$, ton, (*tension*).

2. Determine the nature and amount of the stress, induced in the bar, 14, of the same roof, considered as forming part of the larger secondary truss, *A D C E A* ; and thence deduce the nature and amount of the resultant stress in the same member.

Component Stress = $+\ 3\frac{1}{4}$ tons.
Resultant Stress = $+\ 2$ tons.

3. Find the thrust in bar, 15, of the same structure, due to the loaded larger secondary truss, *A D C E A.*

Thrust = $+\ 2\frac{5}{16}$ tons.

4. Find the thrust in the same bar, arising from the loads on the primary truss, *A E A.*

Thrust = $+\ 7\frac{3}{8}$ tons.

5. Construct the general reciprocal diagram of the *Swansea Station Roof*, Fig. 165, Pl. I. and shew that the resultant thrust along bar, 15, is equal to the graphic sum of the component thrusts, as found in the last two examples.

6. Construct the general reciprocal diagram of the bridge structure carrying the Great Western Railway over the *Feeder*,

near Bristol, Fig. 180, Pl. II.; and find the natures and amounts of the stresses in bars, x, y, z, due to a series of seven static loads, each of 10 tons, concentrated at the joints, B, C, D, E, F, G, H.

$x = +\ 94{\cdot}2$ tons ; $y = -\ 6{\cdot}8$ tons ; $z = -\ 91$ tons.

7. Find the additional stresses produced in the same members of the *Feeder Bridge* by a moving load of 10 tons, instantaneously concentrated at joint, B.

$x = +\ 8{\cdot}4$ tons ; $y = -\ 2{\cdot}8$ tons ; $z = -\ 7$ tons.

8. Determine the superadded stresses, when the same rolling load of 10 tons has reached, and is just passing over joint, C.

$x = +\ 15{\cdot}6$ tons : $y = -\ 5$ tons ; $z = -\ 13$ tons.

9. What are the additional stresses, produced in bars, x, y, and z, of the *Feeder Bridge*, by the same moving load, when it is located at D. ?

$x = +\ 22{\cdot}8$ tons ; $y = -\ 7{\cdot}4$ tons ; $z = -\ 19{\cdot}4$ tons.

10. Find the superadded stresses in the same bars, when the rolling load is passing over joint, E.

$x = +\ 18$ tons ; $y = +\ 3{\cdot}2$ tons ; $z = -\ 19{\cdot}4$ tons.

11. Determine the same stresses when the moving load is at, F.

$x = +\ 13{\cdot}8$ tons ; $y = +\ 2{\cdot}4$ tons ; $z = -\ 15$ tons.

12. Determine the stresses when the rolling load has reached, G.

$x = +\ 9{\cdot}6$ tons ; $y = +\ 1{\cdot}8$ tons ; $z = -\ 10{\cdot}6$ tons.

13. Finally, determine the stresses in the same members of the *Feeder Bridge*, when the ten-ton rolling load is just leaving the bridge at, H.

$$x = +\ 6 \text{ tons}\,;\ y = +\ 1 \text{ ton}\,;\ z = -\ 6{\cdot}6 \text{ tons.}$$

14. Supposing a succession of rolling loads, each of 10 tons, to pass over the *Feeder Bridge*, from left to right; shew that the stress, induced in the oblique member, y, attains a ***maximum***, when the joints, B, C, D, are loaded, and the other joints, E, F, G, H, are unloaded; and find the stress in, y, due to this condition of load.

$$y = -\ 15{\cdot}2 \text{ tons.}$$

15. Shew that in similar circumstances the maximum-stress in the oblique rod, y', due to rolling load, occurs, when joints, D, E, F, G, H, are loaded, and the other joints, B, C, remain unloaded; and find the induced stress under the given conditions of load.

$$y' = -\ 13 \text{ tons.}$$

16. Construct the reciprocal diagram of the roof-structure, *Bristol Goods-Shed*, Fig. 173, Pl. I.; and find the stresses produced in bars, x, y, z, by the given applied loads.

$$x = +\ 18{\cdot}5 \text{ tons}\,;\ y = +\ \tfrac{1}{2} \text{ ton}\,;\ z = -\ 18{\cdot}6 \text{ tons.}$$

17. Construct the reciprocal diagram of the lattice girder, (four systems), forming part of the *Bridge over Avon at Bath*, Fig. 181, Pl. II.; and find the stresses induced in the various members of the structure, under a uniformly distributed load of 10·5 tons at each apex.

PART III.

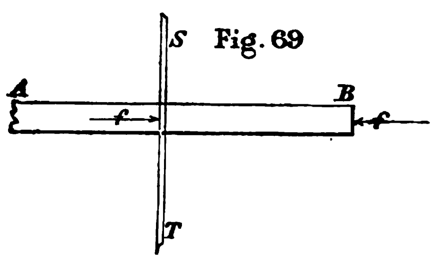
COMPARATIVE STATICS.

CHAPTER I.

DIRECT STRESS.

IF a bar or beam under stress, *A B*, Fig. 69, be divided by a vertical sectional plane, *S T*, the equilibrated condition of the beam demands that a force, *f*, must act at the section, *S T*, equal and opposite to a like force, *f*, acting at the end, *B*, of the beam. Hence, since the position of the sectional plane is quite arbitrary, it may be inferred that the bar suffers a unit stress, $\frac{f}{\Omega}$, in which expression, Ω is taken equal to the area of cross-section.

Fig. 69



Experience has shewn that the elongation taking place in any bar, under the influence of direct stress, varies as its length and the pressure applied to each unit of sectional surface. But, this is true only up to a certain limit of stress. Beyond

that point the elongation ceases to bear a definite proportion to the pressure applied.

If the direction of the force, *f*, Fig. 69, were changed, and it were supposed to act *from* instead of *towards* the centre of the beam; the same intensity of stress would remain, but its nature would change from compression to tension.

The elongation of a bar, that is, its extension under tension, and its contraction or negative elongation under compression, varies *directly*, as before stated, with its own length and the amount of external force applied; *inversely*, however, as its area of cross section and the toughness, or what has been called the elasticity of the material of which it is composed.

Let, E, represent the special toughness or elasticity of the section; then, the results of experience above mentioned can be condensed into the following equational form,

$$\epsilon = \frac{L\,S}{E\,\Omega}; \qquad (1)$$

where, ϵ, represents the elongation; S, the total stress applied; E, the elasticity of the material, Ω, the area of cross-section, and, L, the length of the bar. By a change of form, the same relation gives the value of the toughness of the material, E, in terms of the other factors. Thus,

$$E = \frac{L\,S}{\epsilon\,\Omega} = \frac{S}{\Omega} \div \frac{\epsilon}{L} \qquad (2)$$

Now, $\frac{S}{\Omega}$, expresses the stress per unit of section, and, $\frac{\epsilon}{L}$, the elongation per unit of length; wherefore, the toughness or elasticity of the material is the quotient of the unit of stress by the unit of elongation. This quantity is usually spoken of as the *coefficient of elasticity;* but, since its value varies inversely as the elongation, it would evidently be more rational to term it the *coefficient of toughness.*

By a second change of form given to equation, (1), the unit

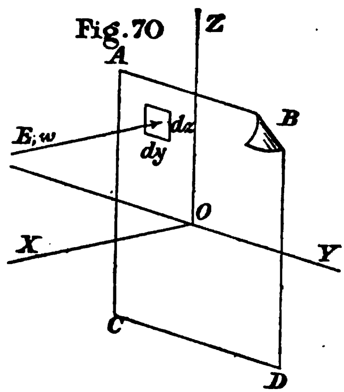
of stress, if unknown, can be calculated in terms of the unit-elongation and the elasticity. Thus,

$$\frac{S}{\Omega} = E\frac{\epsilon}{L} = \frac{s}{\omega} \qquad (3)$$

which relation shews that the unit of stress, $\frac{s}{\omega}$, is equal to the product of the elasticity, E, into the unit-elongation $\frac{\epsilon}{L}$.

The elongation, ϵ, may be positive or negative; or, in more correct terms, the bar may be lengthened or shortened. If it

Fig. 70



be agreed to call extension positive, and contraction of length negative, the stress will take a positive sign, when it stands for tension, and a negative sign for compression.

The unit of stress, $\frac{s}{\omega}$, may, however, vary at different parts of the same cross-section. Instead, therefore, of taking the whole section alone, let a small division of elemental area, ω, be chosen, and let, $\frac{s}{\omega}$, be the unit-stress induced at this part-section. By a relation already given (eq. 3)

$$\frac{s}{\omega} = E_1 . \frac{\epsilon}{L}; \qquad (4)$$

in which expression, E_1, represents the local elasticity of the elemental area considered.

These independent local stresses constitute so many normal forces acting against the surface of cross-section. Their total or algebraic sum will be

$$\Sigma.\, s = \Sigma.\, E_1 \frac{\epsilon}{L}.\, \omega\, ;$$

but, since the whole bar suffers the same unit of elongation, the factor, $\frac{\epsilon}{L}$, will remain constant at all parts of the section; hence,

$$\Sigma.\, s = \frac{\epsilon}{L}.\, \Sigma.\, E_1\, \omega = S \qquad (5)$$

2. CENTRES OF STRESS.—Stresses of the above type, supposed to act parallel to each other, must have a centre through which their resultant acts.

Let, $A\,B\,C\,D$, Fig. 70, represent a cross-section of the beam, and, O, some point in it assumed as origin of co-ordinates, the axes being arranged as shewn in the figure.

From the equations already given it follows that if, E_1, be the local toughness of the material in the vicinity of the elemental area, $d\,y\,d\,z$; $\overline{E_1\,\omega}$, will be a measure of the intensity of the local stress acting at that part of the cross-section [eq. 5].

Remembering that a change in the angularity of a system of parallel forces does not affect the position of the centre of stress, conceive the stresses, $E_1\,\omega$, moved in direction through an angle of 90°, so as to act parallel to the plane, $\overline{Y\,Z}$, Fig. 70. The sum of the moments of the forces so changed, relatively to an axis in the plane, $\overline{Y\,Z}$, will be *nil;* and, since their algebraic sum cannot be equal to zero in all cases, it is to be inferred that their resultant acts in the plane of section.

Again, supposing the forces, *rebatted* into their normal position, so as to act parallel to the plane of, $\overline{X\,Z}$; the sum of their moments about an axis in this plane will be proportionate to,

$$\Sigma.\, E_1\, \omega\, y,$$

the co-ordinate, y, admitting a positive or negative sign, accordingly as it lies on one or other side of the plane, $\overline{X Z}$. The perpendicular distance, y_r, of the centre of stress from the plane, $\overline{X Z}$, will be,

$$y_r = \frac{\Sigma.\ E_1\ \omega.\ y.}{\Sigma.\ E_1\ \omega}$$

Similarly, the distance of the same centre from the plane, $\overline{X Y}$, will be

$$z_r = \frac{\Sigma.\ E_1\ \omega.\ z,}{\Sigma.\ E_1\ \omega}$$

the co-ordinate, z, admitting a positive or negative sign, accordingly as it lies above or below the plane, $\overline{X Y}$.

This centre of stress has been called the *centre of elasticity;* because it coincides with a conventional centre, found by attributing to each elemental area, ω, of cross-section a density proportional to its *coefficient of elasticity.* The values of the co-ordinates of the centre of stress are sufficient evidence of this fact, when they are put into the following forms:—

$$y_r = \frac{\iint E_1\ y\ d\ y\ d\ z}{\iint E_1\ d\ y\ d\ z}\ ;\ z_r = \frac{\iint E_1\ z\ d\ y\ d\ z}{\iint E_1\ d\ y\ d\ z},$$

where, $d\ y\ d\ z$, equals the elemental area, ω, of cross-section.

3. MEASURE OF INEXTENSIBILITY.—If, S, be the total resultant stress applied at one end of a beam ; S, must equal the sum of the parallel stresses induced *at any section* of the beam. Moreover, in order that the bar may be in equilibrium, and that no internal couple may be created, the line of action of, S, must traverse the centres of elasticity of all the sections. Hence, by equation, 5,

$$S = \Sigma.\ s = \frac{\epsilon}{L}.\ \Sigma.\ E_1\ \omega \qquad (6)$$

and by equation, 4,

$$\frac{s}{E_1\ \omega} = \frac{\epsilon}{L}. \qquad (7)$$

From (6),

$$\frac{S}{\Sigma.\, E_1\, \omega} = \frac{\epsilon}{L}; \tag{8}$$

wherefore, by elimination, eqs. (7) and (8)

$$\frac{s}{\omega} = \frac{E_1\, S}{\Sigma.\, E_1\, \omega} \tag{9}$$

It may be inferred, therefore, that for a given tension, S, the unit-elongation, $\frac{\epsilon}{L}$, varies inversely as the expression, $\Sigma.\, E_1\, \omega$ (eq. 8). Hence, this term measures to a certain extent the toughness or *inextensibility* of the beam; further, the unit-tension, $\frac{s}{\omega}$, in any particular fibre, under the general stress, S, and constant inextensibility of section, $\Sigma.\, E_1\, \omega$, is greatest in those parts where the local toughness, E_1, reaches a maximum (eq. 9).

4. EXPERIMENTS ON EXTENSION.—In experiments it is a general rule to consider the material under test to be homogeneous throughout; so that the elasticity, E, is deemed constant, and the general formula of extension given in equation, 5, takes the form,

$$\Sigma.\, s = \frac{\epsilon}{L}.\, E\, \Sigma.\, \omega;$$

that is,

$$S = \frac{\epsilon}{L}.\, E.\, \Omega,$$

whence,

$$E = \frac{S}{\Omega}.\frac{L}{\epsilon}. \quad \text{[See eq. 2.]}$$

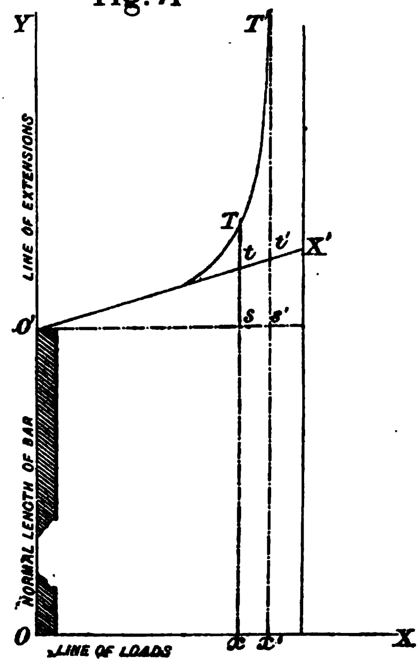
Experimental determinations of the value of, E, are carried out by the help of this equation. A known stress, S, produces a certain elongation, ϵ, in a given length of bar, L, of constant section, Ω. These factors being substituted in the

above equation will determine the value of, E, in the units of measure chosen, which will be of the kind, $\frac{S}{\Omega}$.

In order to ensure success in such tests, certain precautions have to be taken, and amongst others the following :—

1°. The upper supports of the bar must be rigidly fixed; otherwise, they will have an independent elongation of their own, which, being added to the elongation proper to the bar

Fig. 71



itself, will vitiate the results of the experiment. The better method is to make two definite lines on the bar, and note the elongation produced in the interval between them. This will give the value of, ϵ, which, being divided by the length of the interval, will determine the unit of elongation.

2°. The increase of strain must be applied gradually and not by jerks; otherwise impulse will be brought into play, and impair the value of the experiment.

3°. A judicious choice must be made from the many ways existing for the application of stress. The force of gravity; that is, stress applied by weights acting in a vertical sense, is the least perfect of all, and especially where the load required is great, and the resultant elongation of sensible extent.

After a bar has been elongated by means of applied stress, it will tend to return to its original length, when the stress has been removed. But the bar seldom returns to its exact original length. The permanent increase of length, which remains, is called the *set*, and the difference between the total extended length of the bar under stress and its permanent length in *set* is termed *elastic extension*. For example, let distances along the axis, Ox, Fig. 71, represent the total loads, or stresses applied; and ordinates, y, the total extensions during the application of stress. Under these conditions, the ordinate, xT, will give the total extended length of the bar, corresponding to a load, Ox. If the length, Tt, be then taken equal to the *permanent set*, the difference,

$$sT - Tt = st$$

will define the *elastic extension* of the bar. The line or curve, $O'\,T'$, may be termed the curve of *total*, and the line or curve, $O'\,X'$, the curve of *elastic* extension.

The general expression for the elasticity, E, of the material tested is given by the formula,

$$E = \frac{S}{\Omega} \div \frac{\epsilon}{L}, \text{ [eq. 2]},$$

which proves that, E, increases as, ϵ, decreases, and conversely. Therefore, since the elastic extension, st, is always less than the total extension, sT; it follows that the value of, E, is greater when computed for the elastic than for total extension.

The total extension, sT, of a bar rests approximately proportionate to the stress applied, up to a certain limit, after which its value becomes irregular, bearing no longer a strict ratio to the increase of load. This is evident from the form of

the curve of total extension, $O'TT'$, given in the figure. But the curve of elastic extension, $O'X'$, preserves nearly a constant ratio to the increase of load up to the point of rupture. The point where the total extension ceases to bear direct proportion to the load is called the *limit of elasticity* of the material.

The value of the elasticity, E, already defined, has been shewn to depend on two ratios; viz.

$$\frac{\textit{total stress}}{\textit{area of section}} \div \frac{\textit{total elongation}}{\textit{length of bar}};$$

so that, for a bar to have equal elasticity under tension and compression, it is necessary and sufficient that the corresponding ratios, $\frac{\textit{total stress}}{\textit{total extension}}$ and $\frac{\textit{total stress}}{\textit{total contraction}}$, should be equal. The stress and extension in one case may be quite different to the stress and contraction in the other.

The *limit of elasticity* of iron, that is the point at which it ceases to elongate in approximate proportion to the load, varies so greatly in different samples that it would be rash to commit oneself to any rigid rule in the matter. It may be roughly stated to range from 10 to 20 tons per square inch, according to the nature of the material employed. The *breaking load*, or ultimate limit of stress, may be said to vary from 20 to 40 tons per square inch. Higher limits hold for steel, thin iron wire, and other special products. Generally speaking, the limit of elasticity is equal to about half the ultimate limit of stress; though this rule would be misleading in cases where a very high limit of stress obtains.

In like manner, the ultimate extension produced by the breaking load, varies widely for different classes of material. In bar iron it may range from 10 to 30 per cent; whereas in plate iron the limits may be only from 1 to 20 per cent.

The *coefficient of elasticity*, E, is always a large number. It may be stated in any units of measure. In lbs. per square inch cast iron has a coefficient equal to about, 17,000,000; wrought iron, say 29,000,000; and steel, 35,000,000.

Apart from the limits already mentioned, there is another

limit, called the *working limit of stress*, which is generally made equal to about $\frac{1}{6}$th of the ultimate strength of the material; so that a bar, having an ultimate limit equal to 30 tons per square inch, should in fact never be subjected to a strain exceeding 5 tons.

The ultimate limit of stress per unit-area of section of a bar is expressed by the quotient $\frac{S}{\Omega}$, in which, S, represents the total amount of stress applied, when the bar gives way. Similarly, the limit of elastic stress; or, as it is sometimes called, the *proof strain*, is given by the quotient, $\frac{S}{2} \div \Omega$; and the working limit of stress by $\frac{S}{6} \div \Omega$.

Let, f, represent any one of the three quotients,

$$\frac{S}{\Omega}; \frac{\frac{1}{2} S}{\Omega}; \frac{\frac{1}{6} S}{\Omega};$$

the unit-elongation, corresponding to any one of these stresses, will be [eq : 2],

$$\frac{\epsilon}{L} = \frac{f}{E}.$$

Consequently, if the bar be L units in length, its total elongation will be equal in value to

$$\epsilon = \frac{f}{E}. L.$$

Now, the total stress which acts through this elongation of the rod, if gradually applied, is at first zero; and when the limit of stress has been reached, it is $f. \Omega$. Its mean value will, therefore, be $\frac{f \Omega}{2}$. The sum of the work performed in extending the bar by the amount, ϵ, will equal

$$W = \epsilon. \frac{f. \Omega}{2} = \frac{f L}{E} . \frac{f \Omega}{2} = \frac{f^2}{E}. \frac{L \Omega}{2}$$

The coefficient, $\frac{f^2}{E}$, by which the half volume of the bar, $\frac{L\,\Omega}{2}$, is multiplied has been termed the *modulus of resilience* of the bar [Rankine, *Applied Mechanics*, p. 287]. Its value can be expressed in another form; viz., taking, f, equal the ultimate limit of stress,

$$\frac{f^2}{E} = \frac{\left(\frac{S}{\Omega}\right)^2}{\frac{S}{\Omega}\cdot\frac{L}{\epsilon}} = \frac{S}{\Omega}\cdot\frac{\epsilon}{L};$$

in which form it is seen to vary in the compound ratio of the unit-stress and unit-elongation. Hence, this expression is a measure not only of the strength of the bar, but, for applied loads within the limits of elasticity, of the amount, ϵ, through which the bar in part *rebounds*, after having been stretched by a pull or compressed by a thrust.

In the above investigation a mean stress, $\frac{f\,\Omega}{2}$, was taken to act through an elongation, $\frac{f\,L}{E}$, performing work equal to the product of these two factors. This product represents the work done by a force commencing at a zero-value, and gradually increasing up to an intensity, $f\,\Omega$, during the period of extension equivalent to, $\frac{f\,L}{E}$. Hence, the same ultimate elongation will be produced by a sudden pull, $\frac{f\,\Omega}{2}$, and a gradually increased pull, beginning at zero and attaining a maximum, $f\,\Omega$, at the time of rupture. From this fact it may be inferred that the section of a bar must be doubled, if ever it has to withstand a sudden shock equal in amount to the gradually applied stress that would break it.

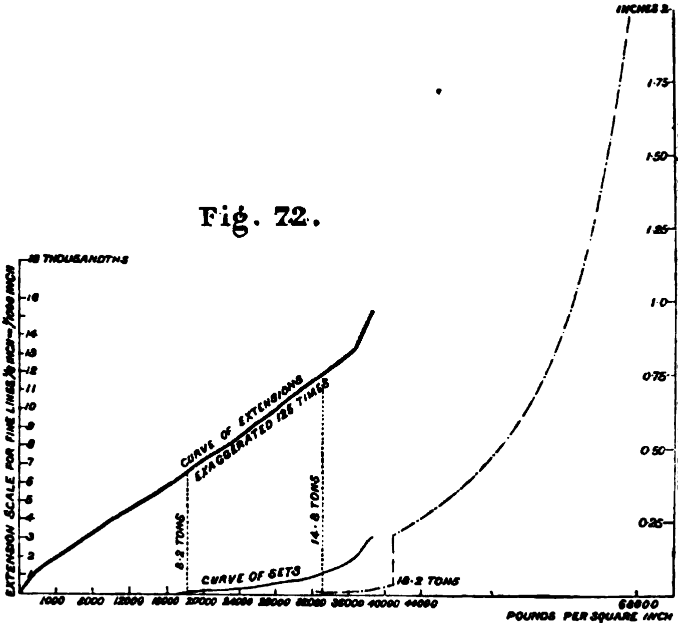
In connexion with the subject of experimental extension, the following extract from a paper, read by Professor Alexander Kennedy before the Institution of Mechanical Engineers, April 1881, will be read with interest.

"In carrying out the experiments presently to be described,

it was thought advisable that a material of the greatest attainable uniformity should be used, and for this purpose Landore '*S. S.*' steel was employed for the plates, and a still milder quality for the rivets. It was next thought important that a careful preliminary investigation should be made of the actual properties of these materials, *i. e.*, the tenacity and extensibility of the plates and rivet steel, and the resistance of the latter to shearing: the effect of punching and drilling upon the plates, both in narrow and broad specimens, was also investigated, and incidentally also the influence of annealing upon the plates. These matters being determined, experiments upon actual joints were made. These included three series. The first consisted of twelve joints, each with two rivets, three different diameters of rivets being used, and with each diameter two proportions of plate and rivet area. The second series consisted of six joints, each with three rivets, of the same diameter, but differently proportioned as to pitch, &c. The results obtained from these were used in the preparation of the last series, consisting of eighteen joints, each having seven rivets, and divided into six sets, each of three similarly proportioned joints. All the joints in these three series were single-riveted lap-joints. The difference between two of the series of experiments upon plates lay only in the method of holding them while being tested;—one series being pulled from pins, the other being held in wedge grips. It may be said that the method of holding does not seem, within the limits of these experiments, to have made any appreciable difference in the strength of the pieces. All these specimens had a length for testing of 10 inches, and both $\frac{1}{4}$ in. and $\frac{3}{8}$ in. were tested in different breadths. The $\frac{1}{4}$ in. plates, when tested $1\frac{3}{8}$ wide, gave an average tenacity of 30·35 tons per square inch, and when tested 4 in. wide, of 30·07 tons per square inch. The mean tenacity of the whole was 30·21 tons per square inch, with 21·2 per cent extension in 10 inches. The $\frac{3}{8}$ in. plate was decidedly milder, having an average tenacity of 28·59 tons per square inch, and an extension of 24·8 per cent in 10 inches. Tested 2 in., wide, its average tenacity was 28·58 tons per square inch,

and tested 3½ inch wide, 28·59 tons per square inch. It appears therefore that within the limits mentioned, no difference is made by alterations of width, so that the width which is most convenient in any particular case may be used with equal certainty of trustworthy results—a fact sometimes worth remembering. The ½ in. plate was tested only in one

Fig. 72.



width, 2¾ in., and had a mean tenacity of 28·96 tons per square inch, and an ultimate extension of 24·8 per cent in 10 inches. It will be seen therefore that the material tested was a very uniform quality of ductile 'ingot iron,' the thin plates being, as was expected, and as was found throughout, somewhat the hardest. The author is informed that the proportion of carbon in these plates, according to analysis at Landore, was about 0·18 per cent.

"Each specimen, before being tested, was scribed across at ½ in. distances throughout its whole length. After fracture the extension was measured first on the whole 10 in. in the

ordinary way, next on the 2½ in. (or quarter length) within which the fracture had actually occurred, and lastly (by subtraction) on the remaining 7½ in. The extension on the 2½ in. nearest to and including the fracture covers practically all of what is usually called the "*local extension*"; and therefore the extension on the remaining part of the length (7½ in.) may be taken as representing the real ultimate extension of the material, or the extension which would be obtained in a test bar so long that the small additional extension close to the fracture did not sensibly affect the whole stretch. In the ¼ in. plate, taking the mean of both series, this last quantity is 16·1 per cent. and in the softer ⅜ in. plate 18·5 per cent. the ½ in. plate giving 17·7 per cent. Somewhat contrary to the author's expectation, the percentages of extension in the 7½ in. are by no means so uniform as those in the whole 10 in., the local extension appearing more or less sensibly to affect the whole of that length. These results emphasise very strongly the well known necessity for specifying always the length of the piece on which a given percentage of elongation has taken place. The extension on the 2½ in. is in one case as high as 53 per cent, and the mean of the ⅜ in. plates is 48·2 per cent; while the extension on 10 in. is only 27 per cent, and on the 7½ in. 19·8 per cent.

"Very detailed observations were made as to the elasticity of the material under test. A simple apparatus was attached to the specimen, which measured the extension, permanent or temporary, between points 10 in. apart. This apparatus neither formed part of, nor touched in any way, the testing machine itself, so that its indications were entirely independent of any strains in the machine, or in any part of the test piece except that lying between the marked points. It was capable of indicating, with very fair certainty, $\frac{1}{10000}$ of an inch. By the use of this apparatus, and the subsequent plotting out of the observations in the form of diagrams, the results given were obtained. By sufficiently careful observation it is possible to distinguish three distinctly marked points in connexion with what might be called the "elastic life" of the material. The first of these is the point at which permanent set begins to be visible. This occurred always at comparatively low loads, far

below the point usually called the limit of elasticity. In one specimen the set curve distinctly commences at a load of 8·21 tons per square inch. Out of the 24 specimens for which this point was determined, it occurred in five cases at less than 9 tons per square inch; and for the whole of the $\frac{1}{4}$ in. and $\frac{3}{8}$ in. plates, it averaged 40 per cent of the breaking load, and about 60 per cent of the load usually called the limit of elasticity. It will be seen (See Figs. 71, 72) that up to a certain point the observed extensions lie upon one straight line with very great exactness; after that point (which in the diagram is reached at 14·78 tons per square inch) the line begins to curve upwards. This second point, the load at which the extension ceases to be uniform, sensibly coincides in two instances with the point at which permanent set first occurs; in all the others it is very much above it. The average set, or permanent extension on removal of load, at the point where uniform extension ends, is about $\frac{4}{10000}$ in. The average total extension in 10 in. for the same point, $\frac{11}{1000}$ of an inch. (See Note).

"Neither of the two points mentioned can be determined without such special and tedious measurements of small extensions as will enable such curves as those of Fig. 72 to be drawn out. Neither, therefore, can be noticed in ordinary testing; and, consequently, neither is the point commonly fixed as the limit of elasticity. If the limit of elasticity be the point at which permanent elongation commences—as it is usually defined to be in books—then its value corresponds to that of the first point mentioned; if, however, it is taken to be the point where the extension ceases to be proportional to the stress (as in Mr. Kirkaldy's valuable experiments on 100 in. bars);—then its value agrees with that of the second point. What is called commercially the limit of elasticity will be found to be a point very considerably higher than the limit which corresponds to any of the usual scientific definitions.

"The second point is certainly the most remarkable point in the life of the material, and it may be worth while to describe the phenomena accompanying it. The testing machine used

Note.—The elastic extension will, therefore be equal to, $\left(\frac{110-4}{10,000}\right)$in. $= \frac{106}{10,000}$ in. [R. H. G.]

was one upon Mr. Kirkaldy's principle, the load being applied by a pump and ram, and the stress in the piece balanced and measured by a movable weight hanging upon a steelyard. So long as the steelyard is floating, (or its free end between the two pins, which limit its motion), the load upon it (multiplied by the proper factor for leverage) is exactly equal to the stress in the piece under test. During the early part of the test the steelyard is kept thus always floating, the increased load, applied by the continued pumping, being continuously balanced by the movement of the weight outwards upon the steelyard. This floating of the steelyard continues long after the loads corresponding to uniform extension are passed, and then suddenly ceases, and without any change in the rate of increase of load, often without the least visible warning, the steelyard drops down and rests on the pin below it. Up to this point the material has been able to balance each increase of load, with only such increase of length as the load itself caused. At this point, however, some structural change appears to occur, which is perhaps best described by the phrase, 'breaking down.' What happens, at least as far as extension is concerned, is shewn distinctly by the dotted line in Fig. 72. For the first part of its length this line is simply a repetition of the black extension line above it, but exaggerated only eight times instead of 500 times. At the point where permanent set appears to begin (8·2 tons per square inch) the total extension was 6·6 thousandths of an inch. It increased uniformly till the stress was 14·8 tons per square inch, and was then 12·2 thousandths, of which 1·2 thousandths was permanent set. The extension then ceased to be uniform, and had increased somewhat rapidly to 33 thousandths, when, at 18·23 tons per square inch, the resistance of the piece suddenly seemed to collapse and break down, and the steelyard dropped in the way described. So far as appearances went, the piece might have continued to balance, say 18 tons per square inch, or even more, for an unlimited time, with the corresponding extension of about 30 thousandths. But once the break-down occurred, the material not only would no longer balance 18·2 tons, but could not even balance a much smaller load. In such a case it will be found on trial (by reduc-

ing the load gradually, and finding the point where the steelyard naturally lifts again of itself) that, with the full extension, the piece will only balance about 80 per cent of the load which has been already upon it, or in this case about 15 tons per square inch. On leaving the weight in the position corresponding to 18·23 tons per square inch, and continuing to pump, it was found that the extension had increased to nearly 200 thousandths, or 0·2 in., before the steelyard lifted again. This sudden increase of extension without increase of load is shewn by the vertical part of the dotted curve in the diagram. After this point, the extension continues to increase faster than the stress, and the curve assumes the well-known appearance shewn.

"Taking averages from the observations, it may be said, in round numbers, that if the extension of the piece, where uniform extension ended, be called, 1, the extension at the point, where the material broke down, would be 4, and would have to increase to, 17, under the same load, before the piece could take any higher load.

"If the limit of elasticity be really taken as the point at which permanent extension begins, it will (for this material) be only 38 per cent. of the breaking load. If it be taken as the latest point where strain and stress seem to be proportionate, it will be about 47 per cent of the breaking load. If lastly, it be taken—as it practically is always for commercial purposes—as the point at which the material 'breaks down;' it is not reached till 68 per cent. of the breaking load.

"The *specific extension* of the material is the actual average extension of the specimen in a length of 10 in., measured in thousandths of an inch for a stress of 1,000 lbs. per square inch. To obtain the actual extension of a piece of the same material under any load, it is only necessary to multiply the specific extension, $\Delta \epsilon$, by the load in thousandths of pounds per square inch, and by the length in inches, and to divide by 10.

"The value of the coefficient or modulus of elasticity, E, is obtained from that of the specific extension; seeing that, E, is

equal to 10,000,000 divided by the specific extension, $\Delta\,\epsilon$." [See Note.]

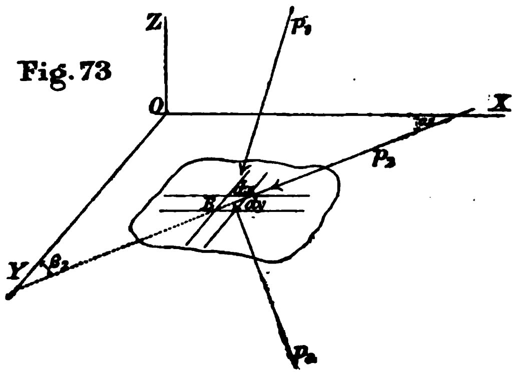
---

5. DISTRIBUTION OF STRESS.—If a uniform stress equal to p lbs. per square inch act normally against a surface containing, Ω, square inches, the resultant stress, expressed in lbs. per square inch, will be,

$$S = p \times \Omega$$

Should, however, the stress be applied at different parts of the surface according to some definite law, the resultant stress must be evaluated in another manner.

Fig. 73



Let the surface, Ω, situate in the plane of $x\,y$, be represented by the curved area given in Fig. 73; and let a small elemental area, E, be enclosed between a pair of lines parallel to the axis of, x, and a second pair parallel to the axis of, y. The area of the whole figure will be equal to

$$\Omega = \iint d\,y\,d\,x.$$

If, moreover, $\phi\,(x\,y)$, represent the local intensity of stress

Note.—By a former equation [§ 1. eq. 2], $E = \frac{S}{\Omega} \div \frac{\epsilon}{L}$. But, in this case, $\frac{S}{\Omega} = 1000$ lbs. per sq. in.; $\epsilon = \Delta\epsilon$, expressed in inches $= \frac{\Delta\epsilon}{1000}$; $L = 10$ in.; so that $E = 1000 \div \frac{\Delta\,\epsilon}{10 \times 1000} = \frac{10,000,000}{\Delta\,\epsilon}$. [R. H. G.]

in the vicinity of the elemental area, E, supposed to be expressible as a function of x, y; the whole stress applied to the element, E, will be,

$$\phi\,(x\,y)\,d\,x\,d\,y,$$

and the whole stress, applied throughout the given surface, Ω, will be

$$S = \iint \phi\,(x\,y)\,d\,y\,d\,x.$$

The mean intensity of stress is defined as the quotient of the stress, S, by the sectional area or surface, Ω, that is by

$$\frac{S}{\Omega} = \frac{\iint \phi\,(x\,y)\,d\,y\,d\,x}{\iint d\,y\,d\,x}.$$

The intensity of stress, $\phi\,(x\,y)$, may be positive or negative. For example, $\phi\,(x\,y)$, might express the height or depth of an ordinate, z, above or below the plane of, $x\,y$; and this ordinate might be directed upwards on one side of the axis of, y, and downwards on the other, as shewn in Fig. 74. The resultant stress would then be found by determining the resultant stress of the positive part, together with its centre of application; and subsequently the same properties for the negative part. The final stress would be equal to the resultant of these separate stresses, looked upon as opposite and, let us say, parallel forces, and the general centre of stress would coincide with that of these two parallel forces.

The determination of the centre of stress has already been considered in its general form. [§ 2.]

Let an element, E, Fig. 73, be subject to a stress, p_2, acting in the plane, x, y; and let the stress at any other part of the given surface be expressible as a function of $(x\,y)$, obeying the same law as, p_2. Suppose that the direction of p_2, makes angles, α_2, β_2, with the axes of x and y. The moment of the stress, p_2, on the element, E, relatively to the axis of, $\overline{O\,Z}$, can

be resolved into two component moments, M_x and M_y. The moment, M_x, will be equal to the product of $E = d\,x\,dy$, the

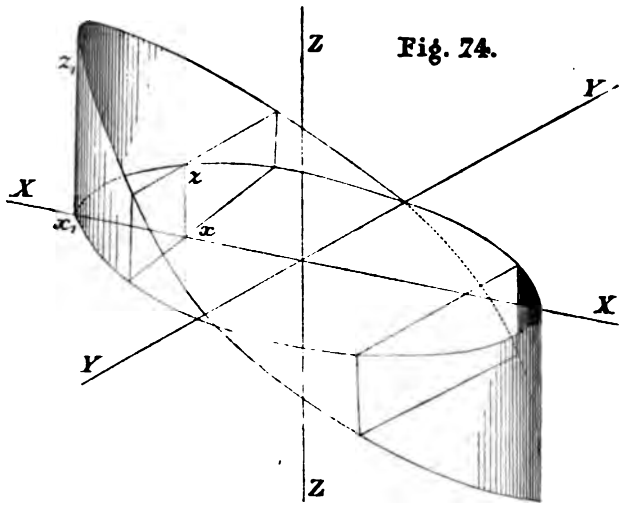


Fig. 74.

intensity of the stress, p_2, resolved perpendicularly to the axis of, x, and the arm-co-ordinate, x; so that,

$$M_x = dy.\,dx.\,p\ sin.\ a_2.\ x.$$

This couple, M_x, will tend to turn the body from left to right about the axis of, z.

Similarly, the moment, M_y, relatively to the axis of, z, is

$$M_y = dy\ dx.\ p_2\ sin.\ \beta_2.\ y,$$

tending to turn the body in the same direction as, M_x. But, if the stress, p_2, were directed towards, instead of away from the origin, O, as happens in the case of, p_3, the resolved moments, M_x and M_y, would differ in sign.

Let, x_0, represent the arm co-ordinate of the resultant moment of a series of parallel forces, p_2; then

$$x_0 \iint p_2\ sin.\ a_2\ dx\ dy = \iint p_2.\ x.\ sin.\ a_2\ dx\ dy\,;$$

hence;

$$x_0 = \frac{\iint p_2 \, x \, dx \, dy}{\iint p_2 . \, dx \, dy}.$$

Similarly,

$$y_0 = \frac{\iint p_2 . \, y . \, dx \, dy}{\iint p_2 \, dx \, dy}.$$

If the intensity be constant at all parts of the surface the symbol, p_2, expressing the general law by which stress varies, can be taken from under the sign of integration, and the co-ordinate arms of the resultant moment will be more simply expressed in the forms,

$$x_0 = \frac{\iint x \, dx \, dy}{\iint dx \, dy}.$$

and,

$$y_0 = \frac{\iint y \, dx \, dy}{\iint dx \, dy}.$$

The term, p_2, which expresses the intensity of stress according to some given law, may be constant, varied uniformly, or accelerated. The case involving constant intensity of stress has just been noticed.

Stress, when uniformly varied, means that the intensity of the stress at any point is directly proportional to the perpendicular distance of that point from a given straight line. For instance, if the straight line be the axis of y, p will vary as $a x$, where, a, equals some constant. In general, if the equation to any straight line, forming the base-line of stress, or the line from which stress varies in proportion to distance, be given by the equation,

$$y = m . \, x + c \, ;$$

and, h, k, be the co-ordinates of any point where stress is applied to the surface, the law of stress can be embodied in the form,

$$p = a\left[\frac{k - mh - c}{\sqrt{1 + m^2}}\right].$$

Accelerated varying stress takes place when p varies as a complex function of x or y, or of both.

Taking an example of uniformly varied stress, suppose the base-line of stress to be the axis of, y. If, in this case, the value of $p = a\,x$, be represented by ordinates, z, Fig. 74, the figure of stress will take the form of a wedge. Further, it will be observed that at any constant distance, x, from the plane of, $\overline{YZ}$, the value of, z, is constant for all points lying in a section passing through, x, and parallel to the plane, $\overline{YZ}$.

Let, z_1, be the intensity of stress for the points situate at a maximum distance from the axis of, y. The extreme ordinate is connected with other ordinates by the following relation,

$$\frac{z_1}{z} = \frac{x_1}{x},$$

from which is deduced

$$z = \frac{z_1}{x_1}.\,x\,;$$

so that by analogy of expression,

$$a = \frac{z_1}{x_1}.$$

Take any element, $dx\,dy$, of the surface, the co-ordinates of which are x, y. The local stress applied to this small area will be equal to

$$a\,x\,dx\,dy\,;$$

and, consequently, the stress applied to the whole surface,

$$S = a\iint x\,dx\,dy.$$

If the above quantity vanish, there is no resultant stress; and it will always vanish, when the line, $\overline{YY}$, traverses the centre of gravity of the figure. For, in that case, the abscissa of the centre of gravity is zero, and the term, $\iint xdxdy$, entering as a numerator into the expression of its value, must therefore vanish.

To take a more general case, let a stress, p_1, make with the axes X, Y, Z, the angles, α, β, γ, respectively [Fig. 73].

Decompose the forces, p_1. applied obliquely to the surface, into three components,

$$p_1 \cos.\alpha \; ; \; p_1 \cos.\beta \; ; \; p_1 \cos.\gamma,$$

parallel to the three axes of co-ordinates, X, Y, Z.

These three components can be re-decomposed into three equal and similar forces acting at the origin, and three series of couples about the axes of x, y, and z, which can be put into the following forms:—

1°. About the axis, $\overline{OZ}$;

$$\begin{aligned} M'_z &= x.\, p_1 \cos.\beta \pm y.\, p_1 \cos.\alpha \\ &= p_1\,(x \cos.\beta \pm y \cos.\alpha) \end{aligned}$$

2°. About the axis, $\overline{OX}$;

$$M'_x = \pm\, y.\, p_1 \cos.\gamma$$

3°. About the axis of, $\overline{OY}$;

$$M'_y = \mp\, x p_1 \cos.\gamma$$

The sign of these moments will depend on what kind of rotation is made positive in calculating the first term of the moment, M'_z.

Summing the above moments for all elemental areas, E, contained in the given surface, and putting, $p_1 = a\,x$; the resultant moments about the different axes will be

1°. About the axis, $\overline{OZ}$;

$$M_z = a\left[\cos.\beta\iint x^2\,dx\,dy \pm \cos.a\iint x\,y\,dx\,dy\right]$$

2°. About the axis, $\overline{OX}$;

$$M_x = \pm\, a\cos.\gamma.\iint x\,y\,dx\,dy$$

3°. About the axis, $\overline{OY}$;

$$M_y = \pm\, a\cos.\gamma\iint x^2\,dx\,dy$$

$$\text{Let, }\iint x^2\,dx\,dy = I\,;\iint x\,y\,dx\,dy = K.$$

The resultant moment, M, will be equal to the square root of the sum of the squares of the above component moments; that is,

$$\begin{aligned} M &= \sqrt{M_x^2 + M_y^2 + M_z^2} \\ &= \sqrt{\begin{aligned} &a^2\cos.^2\gamma.\,K^2 + a^2\cos.^2\gamma.\,I^2 \\ &\qquad + a^2\cos.^2\beta.\,I^2 + a^2\cos.^2 a.\,K^2 \\ &\qquad\qquad \pm 2.\,K.\,I.\,a^2\cos.a\cos.\beta \end{aligned}} \\ &= a\sqrt{\begin{aligned} &[I^2 + K^2]\cos.^2\gamma + I^2\cos.^2\beta + K^2\cos.^2 a \,\pm \\ &\qquad \pm\, 2\,I.\,K\cos.a\cos.\beta \end{aligned}} \end{aligned}$$

Substituting for $\cos.^2\gamma$ its value given by the relation,

$$\cos.^2\gamma = 1 - \cos.^2 a - \cos.^2\beta,$$

the expression for, M, is reduced to the form,

$$M = a\sqrt{I^2\sin.^2 a + K^2\sin.^2\beta \pm 2\,I.\,K\cos.a.\cos.\beta}$$

If, λ, μ, ν, be the angles which the axis of this couple makes, with the co-ordinate axes; then,

$$\cos.\lambda = \frac{Mx}{M}\,;\ \cos.\mu = \frac{My}{M}\,;\ \cos.\nu = \frac{Mz}{M}$$

The following Tables will be found useful in solving the examples annexed to this chapter :—

TABLE I.

HEAT—EXPANSION.

SUBSTANCE.	UNIT-ELONGATION, $\frac{\iota}{l}$, PER DEG. FAHR.	AUTHORITY.
Cast Iron	0·00000617	Roy.
Wrought Iron	0·00000642	Borda.
Brass Rods	0·00001052	Roy.

TABLE II.

COEFFICIENTS OR MODULI OF ELASTICITY.

SUBSTANCE.	COEFFICIENT.		AUTHORITY.
	FROM	TO	
Wrought Iron		29,000,000	Rankine.
Steel Bars	29,000,000	42,000,000	,,
Cast Iron		17,000,000	,,
Cast Brass		9,170,000	,,

EXAMPLES.

1°. If the wrought iron rails of a line of railway were joined together, end to end, for a length of 10 miles, and abutted at the two extremities against immovable supports, what thrust would be exerted at the abutments, in case the temperature of the rails were suddenly raised from 40°, the temperature of fixing, to 96° F. ?

Thrust = 4·65 tons per sq. in.

2°. Let an iron and a brass rod be placed horizontally together and parallel, iron over brass, the interval between the rods being 2 inches, and their lengths equal when the temperature is 60° F. Draw two vertical lines past their two ends, and find points in each of these vertical lines, to which,

when the rods expand differentially, under equal increments of temperature, lines drawn past their two ends continuously and respectively converge.

Common distance = 5·125 ins. above lower brass rod.

3°. If a substance expand under heat by a fraction ϵ of its original length, shew that it increases in bulk by nearly $3.\epsilon$ times its original volume.

4°. Assuming strain to remain proportional to stress, find by what percentage of its length a wrought iron plate, 10 ins. long, 2 ins. wide, and $\frac{1}{2}$ in. thick would extend under a stress of 12·5 tons; and shew that, theoretically, the percentage of extension does not vary with the length of the piece?

Result. 0·0965 per cent.

5°. Suppose a solid wrought iron girder of 20 inches mean section to be accurately bedded between immovable abutments, 100 feet apart, find the horizontal thrust the girder would exert against the abutments, if the temperature were raised from 60° F., registered by thermometer at time of fixture, to 100° F.

Thrust = 66·5 tons.

6°. Supposing the side-rafters of the *Weymouth Goods Shed*, Fig. 166, Pl. I. to be made of wrought iron, and that the scantlings were designed to have *one square inch* of section for every 5 tons of *mean pressure*, due to the given static loads, what thrust would they exert against the principals, if the temperature rose suddenly from 60° F. to 100° F.

Thrust = 7·12 tons.

7°. By how much would one of the side-rafters of the *Weymouth Goods Shed* contract, if it were made of wrought iron, 2·14 sq. ins. scantling, and subjected to the *mean* of the four thrusts acting at different parts of its length.

Contraction = 0·1854 in.

8°. What horizontal play would be required in an expansion-joint at the crown of a wrought-iron arch of 100 feet span, and 10 feet rise, taking the sections as calculated for a working stress of 5 tons per sq. in.; and the temperature to rise and fall 40° relatively to the temperature at the time of erection.

Horizontal play = 0·61632 ins.

9. Determine the work done in the elongation of a wrought-iron bar, 2 in. square, and 40 feet long, when subjected to a stress of 40 tons.

Result. 1344 foot-lbs.

CHAPTER II.

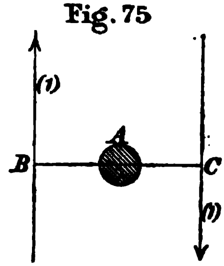
COUPLES OF FORCES.

1. Definition of Couples.—A couple consists of two parallel forces, (1), (1), Fig. 75, equal to each other but opposite in sense, acting through separate points, B and C, between which a perpendicular distance, $\overline{BC}$, intervenes, called the *arm* of the couple. The force of turning which the couple can exert is technically termed its *moment*, equal in the given figure to

$$\text{Intensity of Force (1)} \times \text{arm, } \overline{BC}.$$

The couple is right-handed when it tends to turn the body in the direction followed by the hands of a watch. Consequently, the sense of the couple given in Fig. 75, is right-

Fig. 75



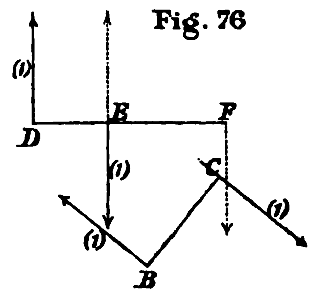
handed; and, if it be agreed upon, positive, understanding by negative couples those which tend to turn the body in the reverse direction.

It is evident that if a second couple existed, parallel and opposite to the one shewn in Fig. 75, and having the same

moment or rotating power, the body would be in equilibrium, provided the couples acted in the same or parallel planes. Further, if the plane of the second couple were obliquely inclined to the plane of the first, the tendency to turn, or power of revolution, of the couples would still be identical in one and the other; though the direction in which they tended to rotate the body would be different. In this case a kind of mixed rotation would be given to the body.

2. ADDITION OF COUPLES.—Let the two couples shewn in Fig. 76, act in the same plane, and let the forces composing them be equal to each other and be represented by (1). The arms, $\overline{DE}$ and $\overline{BC}$, may be of different lengths. The couple, $\overline{BC}$, can be shifted into the position, $\overline{EF}$, without affecting the rotatory

Fig. 76



power it brings to bear upon the body; for, in this new position, it has the same arm, $\overline{EF} = \overline{BC}$, and the same force, (1); so that its moment in one or the other position is expressed by, $(1) \times \overline{BC}$. After transposition it will be seen that the forces, (1) and (1) at E, destroy each other, and the forces, (1) and (1), at D and F, constitute a couple, the moment of which is equal to

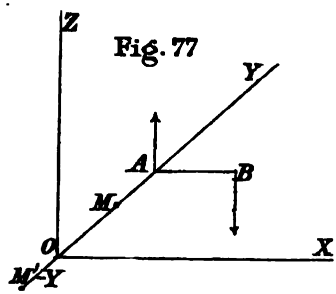
$$(1).\ \overline{DF} = (1).\ [\overline{DE} + \overline{EF}] = (1).\ \overline{DE} + (1).\ \overline{BC};$$

so that, describing the couples by their arms, the two independent couples, $\overline{DE}$ and $\overline{BC}$, can be represented by their sum, $\overline{DF}$. Let two couples, having different arms and forces, be symbolised by $(1).\ \overline{AB}$ and $(2) \times \overline{DC}$. These two couples

can be reduced to forms having the same force, (1). For; let, $\frac{\text{force, (1)}}{\text{force, (2)}} = r$; then, (2) × $\overline{DC}$ can be put in the form

$$(2).\ \overline{DC} = \frac{(1).}{r}\ \overline{DC} = (1).\ \frac{\overline{DC}}{r},$$

in which expression the term, $\frac{\overline{DC}}{r}$, may be named the *reduced arm* of the couple, (2). $\overline{DC}$. On the same principle any number of couples may be reduced to couples having a common force, and the sum of the moments of such a system will be equal to this common force multiplied by the sum of the reduced arms. The sum of these arms, taken with their



proper signs, can be graphically represented by a resultant arm;—hence, it is clear "*that a combination of any number of couples having the same axis is equivalent to a couple whose moment is the algebraic sum of the moments of the combined couples.*" [Rankine, *Applied Mechanics*, § 32.]

Two opposite couples of equal moment acting in parallel planes balance each other; so do two opposite sums of couples, when the moments of the right-handed are together equal to the moments of the left-handed couples of the system. For, all the moments can be expressed in terms of a common force, in which case the sum of the reduced arms will vanish, provided a negative sign be affixed to the arms of the negative couples. Fig. 77, illustrates this statement. The right-handed couple, $\overline{AB}$, is represented graphically by the line, $\overline{OM}$; so that to an observer stationed at, O, the couple appears to turn

the body in the direction followed by the hands of a watch. All right-handed couples acting parallel to the plane, $\overline{ZX}$, would be set off along the axis, $\overline{OM}$, produced beyond, M; and all negative couples along the axis, $O\,M'$; so as to appear *right-handed* to an observer stationed at, O. Now, by the terms of the statement, the sums of the right-handed and left-handed couples are equal; therefore the positive and negative values of, y, corresponding to the axes, $\overline{OM}$ and $\overline{OM'}$, are equal to each other, and there is no tendency to rotation parallel to the plane, $\overline{ZX}$.

3. Composition of Couples.—Let two couples of like sign act upon a body, and after being reduced to the same force, and shifted in planes respectively parallel to those in which they act, let them be represented by the arms, $\overline{OA}$ and $\overline{OB}$, Fig. 78, intersecting at, O. The plane $A\,O\,B$ will be

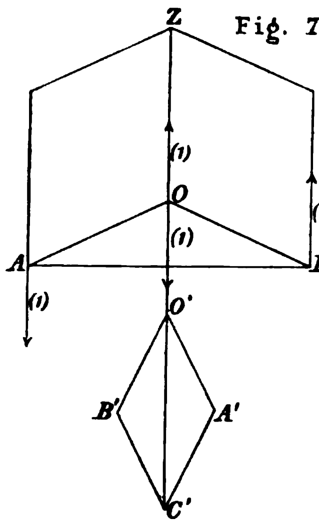


Fig. 78.

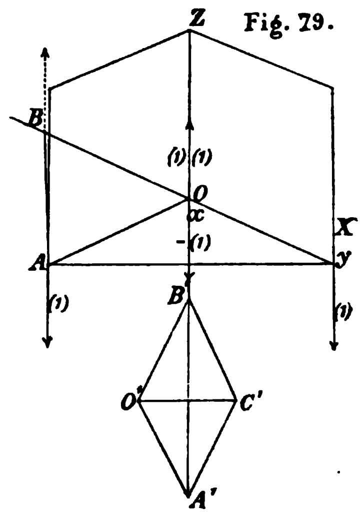


Fig. 79.

perpendicular to the planes of the couples. Since the couples are supposed to have been reduced to the same force, the two equal and opposite forces at, O, balance each other, and the forces at, A and B, remain, constituting a couple equal to, (1) $\overline{AB}$.

From any point, O', in the plane, $A\,O\,B$, draw a line, $O'\,A'$, perpendicularly to line, $\overline{O\,A}$, representing according to any scale the moment of the couple, of which, $\overline{O\,A}$, is the arm. Similarly, draw a line, $O'\,B'$, perpendicularly to, $\overline{O\,B}$, equal to the moment, (1). $\overline{O\,B}$. Let these lines represent the couples, $\overline{O\,A}$ and $\overline{O\,B}$, both in direction and intensity, in such manner that their absolute lengths are representative of the moments, and their directions drawn so that to an observer stationed at, O', looking along, $O'\,A'$ and $O'\,B'$, the couples may appear right-handed. Complete the parallelogram, $O'\,B'\,C'\,A'$; then, the diagonal $O'\,C'$, will represent the resultant couple, $\overline{A\,B}$, both in direction and intensity. For, in the first place, $O'\,C'$, is perpendicular to the arm, $\overline{A\,B}$, and, moreover, by similar triangles,

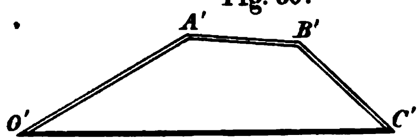
$$O'A' : A'C' = O'B' : O'C' :: \overline{O\,A} : \overline{O\,B} : \overline{A\,B}.$$

If the two couples are of unlike sign, the diagram must be slightly modified. Let the couples be represented by the arms, $\overline{x\,y}$ and $\overline{O\,A}$; $\overline{x\,y}$, being right-handed and, $\overline{O\,A}$, left-handed. Should any doubt exist as to the *handedness* of couples, the planes containing them may be supposed to turn upon their common axis of intersection, $\overline{O\,Z}$, as a hinge, till they form one and the same plane. The couples will then be like or unlike, accordingly as they tend to rotate the body in one or the other direction. The effect of the couple, $\overline{x\,y}$, on the body will not be changed by shifting it into another position, $\overline{O\,B}$, situate in the plane of its action, $\overline{Z\,X}$. In this new position the equal and opposite forces at, O, balance as before, and the couples resolve themselves into a resultant couple, represented by, $\overline{A\,B}$, which is determined in direction and amount by the line $O'\,C'$. The resultant rotation takes place in a plane at right angles to the axis, $O'\,C'$, and in such a direction that to an observer stationed at, O', and looking along, $O'\,C'$, the motion will appear right-handed. Since, $O'\,C'$, Fig. 79, is at right angles to $O'\,C'$, Fig. 78, it may be inferred that, other things remaining constant, a change in the sign of one of the couples has the effect of turning the axis of the

resultant couple through a right angle, provided, as in this case, the moments of the couples are equal.

If there existed a third couple, equal in moment but opposite in sign to the couple, $\overline{AB}$, resultant of the two components, $\overline{xy}$ and $\overline{OA}$, the body would be in equilibrium. In other words, three couples represented in direction and intensity by the three sides of a triangle balance each other. Hence, the *axes* which graphically represent the moments of couples can be treated by the same laws and diagrammatic forms as simple forces. Now, if a series of forces, applied to

Fig. 80.

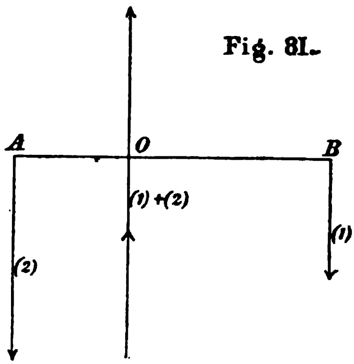


any body, be represented in direction and intensity by the sides of a polygon, the body will be in equilibrium, if the polygon form a closed figure; but if not, the completing line, joining the loose ends of the polygon, will equal the resultant of the forces. Consequently, if in Fig. 80 a series of moments applied to a body be represented in direction and intensity by the successive axes, $O'A'$, $A'B'$, $B'C'$, $C'O'$; the body will be in equilibrium; but, if the axis $C'O'$ be omitted there will be a resultant moment represented in direction and intensity by $O'C'$. It must be observed that, in reading the sense of the moments, it is necessary to commence at the origin, O', and go round the figure in order; that is to say, the first moment is represented in direction and intensity by, $O'A'$; the second by $A'B'$; the third by $B'C'$; and the fourth by $C'O'$; but, should the fourth couple be wanting, the moment resultant of the system will be given by the line, $O'C'$, equal in amount but opposite in sense to the omitted couple, $C'O'$. the addition of which would establish equilibrium.

4. THE PRINCIPLE OF THE LEVER.—Let the couples, $\overline{OA}$ and $\overline{OB}$, acting in the same plane, keep a given body in balance. Shift these two couples in their plane of action till two of the

forces, (1), (2), coincide in direction and pass through the same point, O, Fig. 81. The sum of the two forces, (1), (2), will equal the resultant force applied at, O. Hence, three forces, (1) and (2) acting downwards at A and B, and, (1) + (2), acting upwards at, O, will keep the body in balance. These

Fig. 81.



three forces may be regarded as separate forces; or as equivalent to two couples, $\overline{OA}$ and $\overline{OB}$, equal in moment, but opposite in sign. If they are viewed as separate forces, then each force is proportional to the distance between the other two; so that,

$$(1) : (2) : \overline{(1) + (2)} :: \overline{OA} : \overline{OB} : \overline{OA + OB}.$$

This principle follows from the equality existing between the given couples, from which can be deduced the relations,

$$(1) . \overline{OB} = (2) . \overline{OA}$$

or,

$$\frac{OA}{OB} = \frac{(1)}{(2)}$$

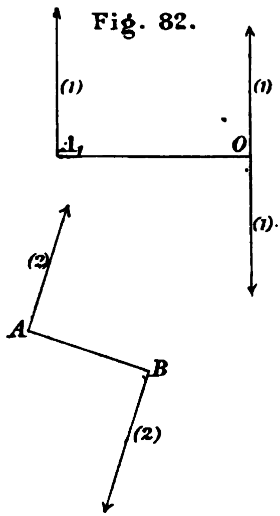
and,

$$\frac{OA + OB}{OB} = \frac{(1) + (2)}{(2)}.$$

5. Composition of Forces with Couples.—Let a force, (1), be applied to a body at a point, O, Fig. 82, and simultaneously let a couple, $\overline{AB}$, acting in the plane of the force or

one parallel to it, be brought to bear upon the same body. Transfer the couple, $\overline{AB}$, making the point, B, coincide with, O, and the reduced arm, $\overline{OA_1}$, equal to, $\frac{(2)}{(1)} . \overline{AB}$. This transposition will not affect the moment of the couple; for, in both cases, it is equal to $(2). \overline{AB}$.

Fig. 82.



Now, the forces, (1) and — (1), at, O, balance each other; so that the original force and couple are equivalent to a single force, (1), acting at, A_1, removed to the left of the original point of application, O, by the amount, $\frac{(2)}{(1)} . \overline{AB}$; equal to the quotient of the couple by the isolated force. Had the applied couple been left-handed instead of right-handed, the point, A_1, would have been shifted to the other side of the origin, O. Generally, it may be stated that right-handed couples shift the points of application of isolated forces to the *left;* whilst left-handed couples shift them to the right, through distances equal to the moment of the couple divided by the isolated force.

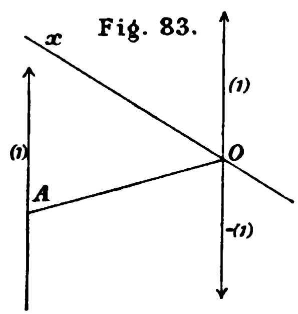
Let an isolated force, (1), Fig. 83, act upon a body through a point of application, A. Let, Ox, be a straight line per-

pendicular to the line of action of the force and not intersecting it. Further, let, $\overline{OA}$, be a line drawn perpendicularly to the force-line, (1), and the axis, Ox.

Through the point, O, apply two opposite forces, (1) and—(1), each equal to the given force. This addition will not interfere with the motion or equilibrium of the body. The isolated force at, A, in conjunction with the negative force at, O, will form a couple, equal in moment to, (1). $\overline{OA}$; and there will remain at, O, a single positive force of the same nature and intensity as the original force, at A.

The moment, (1), $\overline{OA}$, is called the moment of the given force relatively to the axis, Ox; or to the plane which contains it, drawn parallel to the direction of the force.

Fig. 83.



By the foregoing process any number of parallel forces acting in one plane can be resolved into a system of forces, applied at some fixed centre, O, and a system of couples related to some definite axis, Ox. For equilibrium, it is necessary that each of these systems be independently balanced;—the first, in order that there be no movement of translation given to the body;—the second, in order that there may be given no movement of rotation.

If, therefore, upward forces and perpendiculars to the *left* of the plane of, Ox, be called positive; it will be seen that the moment, (1). $\overline{OA}$, Fig 83, is a positive couple, that is, it tends to turn the body in the direction pursued by the hands of a watch. On the other hand, if the same force were applied to the right of the plane of Ox; the arm, $\overline{OA}$, would also

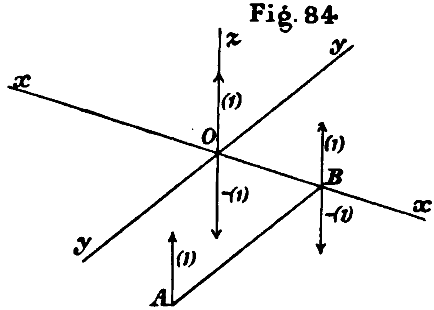
change to the right of the same plane, and would be affected by a negative sign, causing the resulting moment to be negative, with a tendency to turn the body contrary to the motion of the hands of a watch.

If, in the latter case, when the force is transferred relatively to the plane of, Ox, its sign be changed so as to direct it downwards, the resulting moment will still be positive; since in that case it would be expressed by,

$$-(1) \times -\overline{OA} = (1).\overline{OA}.$$

The above conventions are useful in reducing forces to equivalent forces and couples, and the demonstrations given will have made it sufficiently clear that the resultant of any

Fig. 84



number of parallel forces, acting in one plane, is equal to a single force, represented by the algebraical sum of the isolated forces, and that the distance of its point of application from the axis of reduction, Ox, will be,

$$\overline{A_r O_r} = \frac{\Sigma. F. \overline{A O}}{\Sigma. F},$$

in which expression, $\Sigma. F. \overline{A O}$, stands for the algebraical sum of the moments, and, $\Sigma. F$, for the algebraical sum of the forces. When, $\Sigma. F = O$, the system is equivalent to a resultant couple, $\Sigma. F. \overline{A O}$, and when, $\Sigma. F. \overline{A O} = O$, the system is reduced to a resultant force, $\Sigma. F$, acting through the origin.

When both these expressions vanish, the system is in equilibrium.

Suppose it were required to find the moments of a force with respect to two axes; that is, to decompose it into two component moments and an equivalent force. Let the axes about which moments are taken be, $\overline{x\,x}$ and $\overline{y\,y}$, Fig. 84. Draw a line, $\overline{A\,B}$, perpendicularly to the axis, $x\,x$, and the line of action of the force, (1), applied at A. The force (1) at A, is then seen to be equivalent to the couple, (1). $\overline{A\,B}$, and a force, (1), applied at, B. Redecompose the force, (1), at, B, into an equal and similar force at, O, and a couple, (1) × $-\,\overline{O\,B}$, where, $\overline{O\,B}$, is affected by a minus sign, because it lies to the right of the plane, $\overline{y\,z}$, relatively to which it forms a couple-arm.

The original force, (1), at A, can in this way be shewn to be equivalent to an equal and similar force, (1), at, O; and two couples, one positive and equal to, (1). $\overline{A\,B}$, about, $\overline{x\,x}$, as an axis, the other, $-$ (1). $\overline{O\,B}$, negative and having, $\overline{y\,y}$, as its axis of revolution.

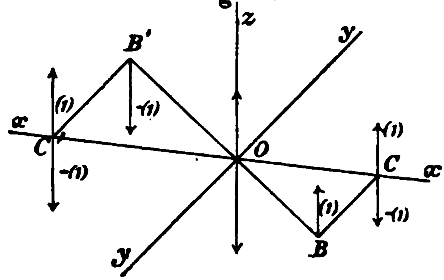
It is very necessary to distinguish the signs of couples. Thus, it will be seen that with the axes given in Fig. 84, al positive or upward forces create positive moments about the axis of, x, when their arms lie *on the left* of the plane, $\overline{z\,x}$. On the other hand, the same forces will give negative couples about the axis of, x, when their arms lie on the right of the same plane.

Hence, it may be inferred that any system of parallel forces acting in different planes is in equilibrium; provided; 1°. that the algebraic sum of the forces be zero. 2°. that the algebraic sums of the moments of the forces relatively to a pair of axes at right angles to each other, and to the common line of action of the forces, be zero. The first of these conditions may be expressed as, $\Sigma.F = O$, and implies the absence of rectilinear movement of translation. The second condition may be put in the form, $\Sigma.y.F = O$, and $\Sigma.x F = O$, which expresses the absence of rotation about the co-ordinate axes; x and y.

Let there be given two equal and opposite forces, (1) and

$-(1)$, Fig. 85, acting at points, B and B' respectively. Decompose each of these forces into a system of forces applied at the origin, O; and a double system of moments about the axes, $\overline{xx}$ and $\overline{yy}$.

Fig. 85.



Since the system of forces at the origin is equal to the algebraical sum of the forces, it follows that

$$\Sigma . F = (1) - (1) = O.$$

The system is, therefore, reduced to a resultant moment which can easily be determined. Decompose the positive upward force at, B, into a moment, $(1)\ \overline{BC}$, about, $\overline{xx}$, and a moment, $(1).\!-\!\overline{OC}$, about the axis, $\overline{yy}$; so that the force, (1), at B, is equivalent to

1° A positive force, (1), acting upwards at, O;
2° A moment, $(1).\ \overline{BC}$, about the axis, $\overline{xx}$;
3° A moment, $(1).-\overline{OC}$, about the axis, $\overline{yy}$.

Similarly the negative, downward force, $-(1)$, at, B', can be resolved into

1° A negative force, $-(1)$, acting downwards at, O;
2° A moment, $-(1).-\overline{B'C'} = (1).\overline{B'C'}$, about, $\overline{xx}$;
3° A moment, $-(1).\overline{OC'}$, about the axis, $\overline{yy}$.

The forces at, O, balance each other, according to what has been shewn.

The algebraical sums of the moments about the axes, $\overline{y\,y}$, and, $\overline{x\,x}$, will be

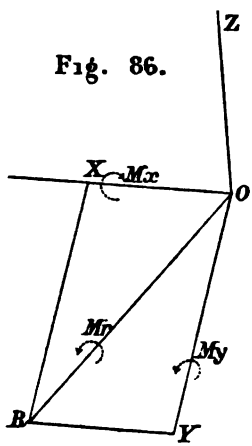
$$\Sigma . F . y = M_x = (1) . \overline{B\,C} + (1) . \overline{B'\,C'} = (1)\,[B\,C + B'\,C']$$
$$\Sigma . F . x = M_y = (1) . -\overline{O\,C} + -(1) . O\,C'$$
$$= -(1)\,[\overline{O\,C} + \overline{O\,C'}] = -(1) . \overline{C\,C'}.$$

Set off along the axis of, x, the graphic value of the resultant moment about that axis,

$$M_x = (1)\,[\overline{B\,C} + \overline{B'\,C'}]\,;.$$

and in such a direction that to an observer stationed at, O, Fig. 86, the couple shall appear right-handed.

Fig. 86.



Similarly, set off on the axis of, y, a line graphically representing the moment, M_y. Complete the parallelogram, $O\,X\,R\,Y$, and draw the diagonal, $\overline{O\,R}$, which will represent in direction and intensity the moment-resultant of the components, $\overline{O\,X}$ and $\overline{O\,Y}$; or M_x and M_y. In other words, the length of the axis, $\overline{O\,R}$, will measure the rotatory power of the resultant couple, and its direction will be determined by the fact that it must appear right-handed to an observer stationed at, O, and looking along the line, $\overline{O\,R}$.

EXAMPLES.

1. If a rectangular sluice-gate, 12 feet wide and 10 feet high, were swung upon an excentric pivot, placed at a distance of $\frac{1}{10}$th of a foot from the vertical line through the centre of the gate, what is the turning power upon the sluice-gate of a column of water, bearing upon two-thirds its height, and moving at a velocity equivalent to 100 lbs. pressure per sq. ft.?

Result. 720 foot-lbs.

2. What power would a man have to exert, to keep the same sluice closed, if he attached a rope to the end of the longer side of the gate, and pulled at right angles to its length.

Result. 118·03 lbs.

3. And if the man pull at an angle to the length of the sluice, at what angle, x^o, will his strength be just sufficient to balance the turning power of the water?

$$x^o = 79^\circ\ 36'\ 47''.$$

4. If I push a barrel over a horizontal distance of 6 feet, pressing down upon it with a pressure of 100 lbs., at an angle of 60° to the horizon, what is the amount of work I perform, and what turning power do I exert upon the barrel, supposing it to be 3 feet in diameter.

Work = 300 foot-lbs.
Turning Power = 150 foot-lbs.

5. Shew, that, in the preceding example, the work done in turning the barrel through an arc, equal in length to the radius of its cross-section, is one half that required to set it in motion.

6. Determine the true weight of a body, W, when weighed in a false balance, the arms of which, instead of being equal, are of different lengths, the longer measuring, a feet, the

shorter, b feet; and the weight of the body being, P lbs., when placed in the scale attached to the long arm, and, p, lbs., when placed in the other scale.

$$W = \sqrt{P \cdot p}$$

7. Shew that the tractive power necessary to carry the wheel of a locomotive over a sleeper, maliciously laid across its path, varies, other things being equal, inversely as the square root of the radii of the wheels.

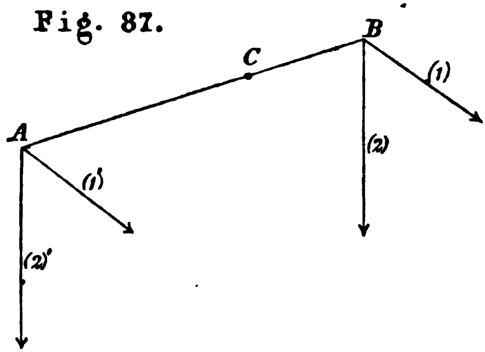
CHAPTER III.

RESULTANTS AND CENTRES OF FORCE.

1. PARALLEL FORCES IN ONE PLANE.—Let a pair of parallel forces, (1), (1′), be applied at the points, A and B, Fig. 87, and join these points by a straight line, $\overline{AB}$. In this line let a point, C, be taken such that

$$\frac{AC}{BC} = \frac{(1)}{(1')}$$

Fig. 87.



It follows, according to the principle of the lever [Pt. III. Ch. II. § 4] that the point, C, must lie in the line of action of the resultant of the forces, (1) and (1′). This point is sometimes called the centre of parallel forces. Its position does not depend on the actual magnitude of the forces; for it is clear that the numerator and denominator of the above ratio can

be multiplied by any constant factor, m, without involving any change in its absolute value. Thus,

$$\frac{AC}{BC} = \frac{(1)}{(1')} = \frac{m.(1)}{m.(1')}$$

In like manner the forces need not have a constant inclination, so long as they remain parallel to each other, which can be proved by making, $m = \cos.\theta$, θ being the angle between the new and old directions of the forces.

Let one force of a parallel system act through a point, A, Fig. 88, and let this point be referred to three co-ordinate axes, $\overline{OX}$, $\overline{OY}$, $\overline{OZ}$. The magnitude of the force at, A, or any quantity proportionate to it may be denoted by, F.

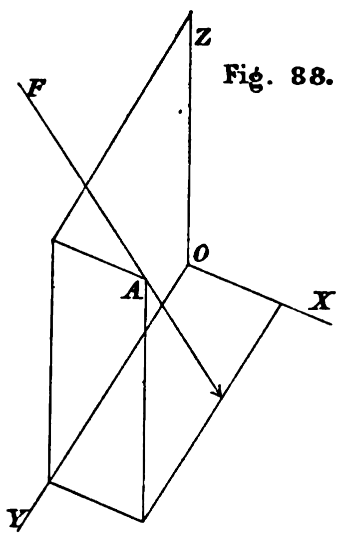


Fig. 88.

In order to find the centre of such a system of parallel forces, it is permissible to vary the common angularity of the forces. Let them act parallel to the plane, $\overline{YZ}$. The sum of the moments of the forces relatively to that plane will be, $\Sigma.xF$. Hence, the distance of the resultant of parallel forces from the plane, $\overline{YZ}$, will be

$$x_r = \frac{\Sigma.xF}{\Sigma.F}$$

Likewise, its distances from the planes, $\overline{ZX}$, and, $\overline{XY}$, are given by

$$y_r = \frac{\Sigma . y F}{\Sigma . F}$$

$$z_r = \frac{\Sigma . z F}{\Sigma . F}$$

The co-ordinates, x_r, y_r, z_r, determine the centre of parallel forces of the given system; that is, a point through which their resultant passes.

2. Forces not parallel in one Plane.—Let a system of forces having different directions act in one plane. Choose an axis perpendicular to the plane of action of the forces, and piercing it in a point, O, Fig. 89. Resolve the forces into an equivalent system of forces applied at, O; and a corresponding

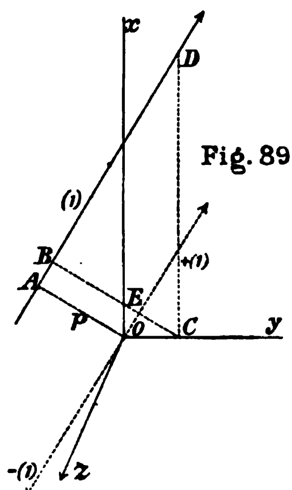


Fig. 89.

system of couples [Pt. III. Ch. II. § 5]. For instance, force, (1), can be represented by an equal and similar force applied at, O; and a corresponding couple, $(1).\overline{AO}$, where, $\overline{AO}$, is the perpendicular distance from, O, to the line of action of the force, (1).

Let, $\overline{A_r O_r}$, be the perpendicular distance from, O, of the resultant force of the system, the direction of this perpendicular being determined by that of the resultant, R, of the

transferred system applied at, O, with which it must form a right angle. Let, $\Sigma. M$, be the sum of the corresponding moments of the type, $(1). \overline{A\,O}$. It follows that

$$\overline{A_r\,O_r}.\,R = \Sigma.\,M$$

or

$$\overline{A_r\,O_r} = \frac{\Sigma.\,M}{R}$$

If it be agreed to call arms on the left of the point, O, positive, and those on the right negative, allowing upward forces to be positive and downward forces negative, the direction of $\overline{A_r\,O_r}$ will be determined by the sign of the quotient, $\frac{\Sigma.\,M}{R}$.

It is, however, much simpler to treat this case by graphic statics.

Construct the polygon of forces at, O, corresponding to the transferred system of forces; and subsequently the polar polygon of the forces by the following method :—

Commencing at the origin, O, Fig. 91, draw the polygon of

Fig. 90.

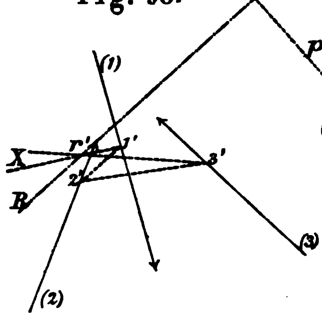
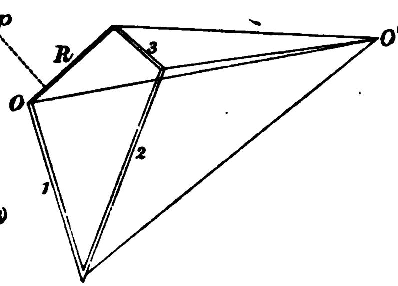


Fig. 91.



forces, shewn in double lines, which corresponds to the system, (1), (2), (3) ; the objective paths of these forces being indicated by arrow-headed lines on the left of the same figure.

Pitch any pole, O', and draw polar lines to the corners of the polygon of forces. From any arbitrarily chosen point, X, in

the plane of forces, draw the line $\overline{X\,1'}$, parallel to, $\overline{O\,O'}$, intersecting force-line, (1), in, 1'. From this point draw a second line, 1' 2', parallel to, $O'_{1\,2}$, [$O'_{1\,2}$ is a convenient symbol for the line connecting the pole, O', with the junction of lines, 1 and 2, on the polygon of forces]. From, 2', draw a line, 2' 3', parallel to, $O'_{2\,3}$, and from, 3', a line, 3' r', parallel to polar line, $O'_{3\,R}$. The line, 3' r', will intersect the line, $X\,1'$, first drawn, in a point, r', which determines an objective point on the path of the resultant force, R, the direction and intensity of the same force being given by the shaded line in the polygon of forces. The knowledge of these three properties of the resultant force, R; viz., its direction, intensity, and a point in its line of action, is all that is necessary for its full definition.

The resultant, R, so found, can be decomposed into an equal and similar force, R, applied at, O, and a corresponding couple $R.\,p$; where, p, is the perpendicular distance from, O, on the objective path of, R.

If, for any given system of forces, having any directions but acting in one plane, the polar polygon and the polygon of forces both form closed figures, it is sufficient evidence that the system is in equilibrium; if, however, the polygon of forces close, and not the polar polygon, the system reduces to a couple equal to the algebraic sum of the moments of the forces separately taken. Should the point, r', coincide with, O, there is no moment existing about that point, the system being then equivalent to the resultant force, R, acting at, O.

Another solution of the same problem can be obtained by means of a system of rectangular co-ordinates. Let, $\overline{x\,y}$ Fig. 89, be the plane of inclined forces, and, $F = (1)$, any force of the given system acting in that plane. Let a point, O, where an axis, Oz, perpendicular to the plane of forces, pierces its surface, be taken as the origin of co-ordinates. Suppose, Ox, the vertical axis, and, Oy, the corresponding horizontal axis of co-ordinates. Finally, let the force, F, make an angle, a, on the right of its direction with the axis of, x.

Resolving the force, F, along the axes of x, and y, we obtain,

$$F_x = F\cos.a\,;\ F_y = F\sin.a.$$

The rectangular components of all the forces may, therefore, be represented by

$$\Sigma.\ F_x \text{ along the axis of, } x, = \Sigma.\ F\ cos.\ a$$
$$\Sigma.\ F_y \text{ along the axis of. } y, = \Sigma.\ F\ sin.\ a.$$

The magnitude of the resultant, R, will be

$$R = \sqrt{(\Sigma.\ F_x)^2 + (\Sigma.\ F_y)^2},$$

and the angle, a_r, which it makes with the axis of x, on the right of its own direction, will be determined by the relations,

$$cos.\ a_r = \frac{\Sigma.\ F_x}{R}\ ;\ sin.\ a_r = \frac{\Sigma.\ F_y}{R}$$

The signs of $\Sigma. F_x$, and $\Sigma. F_y$, will determine the direction of, R. It remains to find the perpendicular distance from, O, of its objective path. The distance from, O, of any one of the forces, F, will be

$$\begin{aligned} \overline{AO} = p &= \overline{BC} - \overline{EC} \\ &= \overline{CD}.\ sin.\ a - \overline{OC}\ cos.\ a \\ &= \quad x\ sin.\ a - y\ cos.\ a, \end{aligned}$$

in which equation, x and y, are the co-ordinates of any point, D, on the line of action of the force. It will be seen that, in agreement with a convention already established, the arm, p, is positive in this case. [Pt. III. Ch. II. § 5.]

The moment of the force, F, relatively to the axis, Oz, is

$$F.\ \overline{AO} = F.\ [x\ sin.\ a - y\ cos.\ a],$$

and the sum of all such moments relatively to the same axis,

$$\Sigma.\ F.\ [x\ sin.\ a - y\ cos.\ a] = \Sigma.\ F_y\, x - \Sigma.\ F_x\, y.$$

The perpendicular distance from, O, of the path of the resultant force will, therefore, be equal to

$$A_r O_r = \frac{\Sigma.\ F_y x - \Sigma.\ F_x y}{R}$$

Let, x_r, y_r, be the co-ordinates of any point on the line of action of, R; and, a_r, the angle its direction forms with the axis of x. Representing $\overline{A_r\, O_r}$ by, p_r, the equation to the path of, R, will take the form,

$$p_r = x_r \sin. a_r - y \cos. a_r.$$

But,

$$\sin. a_r = \frac{\Sigma. F_y}{R}; \cos. a_r = \frac{\Sigma. F_x}{R}.$$

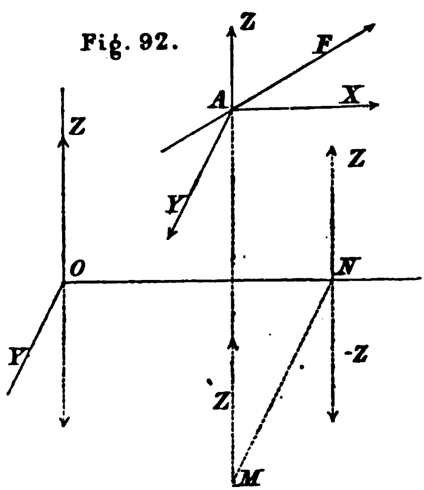
Hence,

$$R. p_r = x_r \Sigma. F_y - y_r \Sigma. F_x.$$

If the sum of the moments, $\Sigma. F. \overline{A\,O}$, vanish, the resultant acts through the origin. If, $R = O$, the system is equivalent to the couple. $\Sigma. F. \overline{A\,O}$.

3. FORCES NOT PARALLEL IN DIFFERENT PLANES.—Let, F, Fig. 92, be one force of a system acting in different planes

Fig. 92.



and in various directions, and let x, y, z, be the co-ordinates of the point of application of the force, F. Resolve, F, into three component forces, X, Y, Z, parallel to the co-ordinate axes. Suppose the component, Z, when produced in direction, to meet the plane of, $x\,y$, in a point, M. Draw the line, $\overline{M\,N}$,

parallel to the axis, $\overline{OY}$, intersecting, $\overline{OX}$, in N; and at, N, apply two opposing forces equal to each other and to the component, Z.

It will be evident that the part-force, Z, applied at, M, is equivalent to a force, Z, at N, identical in amount and direction to, Z; and to a couple, $Z \times \overline{MN} = Z.y$, about the axis of, x.

Again the force, $+Z$, applied at N, is equivalent to an equal and similar force, $+Z$, at O, together with a couple, $Z \times -\overline{ON} = -(Z.x)$, about the axis of y; so that, finally, the force, Z, applied at, A, is equivalent to

1° An equal and similar force, Z, at O;
2° A positive couple, $Z.y$, about the axis of x;
3° A negative couple, $-Z.x$, about the axis of y;

In like manner the force, Y, is equivalent to

1° An equal and similar force, Y, applied at O;
2° A couple, Yx, about the axis of, z;
3° A couple, $-Yz$, about the axis of, x;

Thirdly, the force, X, is equivalent to

1° A force, X, at, O;
2° A couple, Xz, about the axis of, y;
3° A couple, $-Xy$, about the axis of, z;

Hence, the original force, F, can be resolved into a series of forces, X, Y, Z, applied at the origin, O, along the axis of co-ordinates; and a series of couples of which the moments will be

$Z.y - Yz$, about the axis of, x;
$X.z - Zx$, about the axis of, y;
$Yx - Xy$, about the axis of, z.

Decomposing the other forces, F_2; F_3, &c., into similar series, it will be seen that the given system is equivalent to a

series of forces directed along the co-ordinate axes, which may be represented by

1° $\Sigma. X$, along the axis of, x;
2° $\Sigma. Y$, along the axis of, y;
3° $\Sigma. Z$, along the axis of, z;

together with the series of couples,

1° $\Sigma\,[Z.y - Yz]$, about the axis of, x;
2° $\Sigma\,[Xz - Zx]$, about the axis of, y;
3° $\Sigma\,[Yx - Xy]$, about the axis of, z.

Let, R, be the resultant force of the system, and α, β, γ, the angles its direction makes with the co-ordinates, x, y, z; then

$$R^2 = \overline{\Sigma X}^2 + \overline{\Sigma. Y}^2 + \overline{\Sigma. Z}^2;$$
$$\cos. \alpha = \frac{\overline{\Sigma. X}}{R}; \quad \cos. \beta = \frac{\overline{\Sigma Y}}{R}; \quad \cos. \gamma = \frac{\overline{\Sigma. Z}}{R}.$$

Let, M_r, be the moment-resultant of the three series of couples given above; that is, of, M_x, about the axis of x; M_y, about the axis of, y; and of M_z about the axis of, z; then

$$M_r^2 = M^2_x + M^2_y + M^2_z$$

If, moreover, λ, μ, ν, represent the angles which the *axis* of this resultant moment makes with the co-ordinate axes,

$$\cos. \lambda = \frac{M_x}{M_r}; \quad \cos. \mu = \frac{M_y}{M_r}; \quad \cos. \nu = \frac{M_z}{M_r}.$$

The conditions for equilibrium are six; viz.,

$$\overline{\Sigma. X} = 0; \quad \overline{\Sigma. Y} = 0; \quad \overline{\Sigma. Z} = 0;$$

and,

$$M_x = 0; \quad M_y = 0; \quad M_z = 0.$$

The first three equations express the absence of rectilineal movement along the axes; the second three exclude all movement of rotation about the same lines. When the system

is not in equilibrium, the several cases which may arise must be separately considered.

Case I. If $M_r = O$, and, R, exist, the resultant acts through the origin, and the movement is entirely one of translation.

Case II. When the axis of, M_r, is at right angles to the direction of, R;—a case expressed by either of the two conditions

$$\cos. a. \cos. \lambda + \cos. \beta. \cos. \mu + \cos. \gamma. \cos. \nu = o,$$
$$\overline{\Sigma. X}. M_x + \overline{\Sigma. Y}. M_y + \overline{\Sigma Z}. M_z = O;$$

the resultant of, R and M_r, will be a single force equal and parallel to, R, having its line of action removed from, O, by a perpendicular distance, (that is, perpendicular to the direction of, R, applied at the origin) equal to

$$p = \frac{M_r}{R}.$$

If the couple, M_r, be right-handed, the distance, p, must be set off to the left of the direction of, R; and to the right if the couple be left-handed [Pt. III. Ch. II. § 5].

Case III. When, $R = O$, there remains only the couple, M_r.

Case IV. When the couple, M_r, acts in a plane at right angles to the line of action of the force, R, a condition of things expressed by the relations,

$$\lambda = a;\ \mu = \beta;\ \nu = \gamma;$$

or,

$$\lambda = -a;\ \mu = -\beta;\ \nu = -\gamma;$$

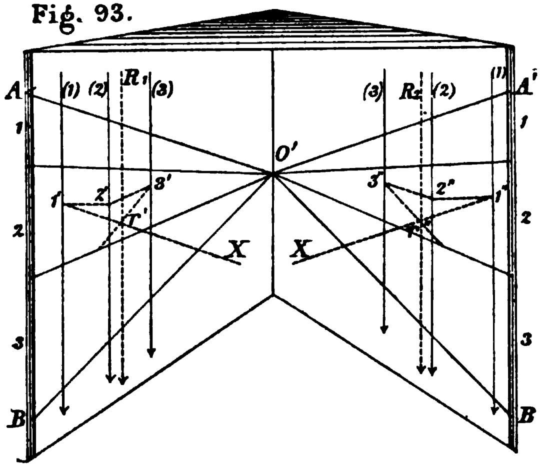
the rotation of the body is quite independent of the direction of R; since it revolves about an axis parallel to the linear direction of this force. A cylinder made to turn about its own axis, and dragged along by the influence of a force parallel to that axis, exemplifies the case in point.

Case V. When the axis of, M_r, is oblique to the direction of, R; it will form with it an angle given by the equation,

$$\cos.\ \theta = \cos.\ a \cos.\ \lambda + \cos.\ \beta \cos.\ \mu + \cos.\ \gamma.\ \cos.\ \nu$$

The couple, M_r, which is represented in moment and direction by a length of its own axis, can be resolved into two components; viz., $M_r \sin.\ \theta$, and $M_r \cos.\ \theta$. The couple, $M_r \sin.\ \theta$, acts in the same plane as, or in a plane parallel to the direction of, the force, R, and can be, therefore, compounded with it, the resultant being a single force, equal and similar to, R, distant from the origin by the length of the co-ordinate-arm, $\frac{M_r \sin.\ \theta}{R}$ [Pt. III. Ch. II. § 5]. The partial couple, $M_r \cos.\ \theta$, induces a movement of rotation about an axis parallel to the direction of, R.

Fig. 93.



4. Graphic Methods.—The intensity and direction of the resultant force of a system can be determined in another way by the methods of graphic statics. For example, let any system of parallel forces be projected upon two planes, parallel to the common direction of the forces. Under these conditions the series of forces will be projected upon the two planes in their full proportions, as shewn in, Fig 93.

Next, find by means of the polygons of forces [lines of forces, $A\,B$ and $A'\,B'$], and the polar polygons, $1'\,2'\,3'\,r'$, $1''\,2''\,3''\,r''$, points, r'; r'', on the projected paths of the resultant force relatively to each plane. Through, r', r'', draw lines, R_1; R_2, parallel to the common direction of the forces; and through those lines, considered as traces of planes, let there be passed two planes making with the planes of projection angles equal to those made by the series of planes, used in the projection of the series of forees, 1, 2, 3. The planes passing through, R_1 and R_2, will intersect in a line, determining the objective path of the resultant force. The magnitude of the resultant is given by either of the lines of forces, identical in form and shewn on the planes of projection.

In a somewhat similar way could be determined the objective path of *any* system of forces acting upon a rigid body. The process, which, however, is not very practical, may be briefly described.

Choose any three planes at right angles to each other, and project the force-lines of the system upon those planes. Construct in space the *gauche* polygon of forces, and project it also upon each of the co-ordinate planes. Find, by means of this projected polygon of forces, a point on the projected line of action of the resultant force;—which can be accomplished by drawing the polar polygon relatively to the projected system of forces on each plane. Let points so found, corresponding to the three planes of projection, be represented by, r_1, r_2, r_3. Through each of these points, draw a line parallel to the *projected direction* of the resultant-force, as given by the line completing the projected polygon of forces on each plane; and through each of the three lines so found, shewing the projected paths of the resultant force, let there be passed three planes, making with the co-ordinate planes the same angles as were made by the planes used to project the force-lines. These three planes will combine by their intersections to determine a point on the *objective path* of the resultant force. Its direction and magnitude will be given by the *gauche* polygon of forces.

EXAMPLES.

1. Two forces equal respectively to 12 tons and 10 tons act upon a particle, at right angles to each other; required their resultant.

Result. 20 tons.

2. Let a tree, $O y_3$, Fig. 182, Pl. I., be pulled by three ropes, one attached to the top, y_3, 42 ft., the other at, y_2, $31\frac{1}{2}$ ft., and the third at, y_1, 23 ft. above ground-level. Let the directions of the ropes be those given in the figure, each man being supposed to haul on a line with his head. Let the strength of the first man be 120 lbs., that of the second 140 lbs., and the united strengths of the two men at the third rope, 200 lbs.;—find the resultant pull, R, and the lengths of the two intercepts, $O X$ and $O Y$, cut off by its direction on the ground line, and axis of the tree respectively.

$$R = 462 \text{ lbs.}; \ O X = 66 \text{ feet}; \ O Y = 30 \text{ feet}.$$

3. Under the same conditions, shew that the combined pulls of the four men are equivalent to a force, $R = 462$ lbs., passing through the foot of the tree at, O, and a moment, $M = 12474$, foot-lbs., about the same point.

4. Taking the same tree, acted upon by the same forces, and supposing it to be held down by two main roots, R_4 and R_5, branching from, O, in the given directions; find the natures and amounts of the stresses induced along the roots.

Thrust, $R_4 = 1565$, lbs.
Tension, $R_5 = 1865$, lbs.

5. Again, assuming that the earth above the roots, which are of the lengths, $O B = 25\frac{1}{2}$ feet, $O D = 19\frac{1}{2}$ feet, offers a passive resistance, respectively normal to the lengths of the roots, and at each point proportionate to the depth of the earth between the ground-level and the point; find the resultant earth-pressure, R_1, and its intercepts on the axes,

which will suffice to equilibrate the combined efforts of the men.

$$R_1 = 891 \text{ lbs.}; \; - O X_1 = 16.2 \text{ feet}; \; - O Y_1 = 33 \text{ feet.}$$

6. Let a ship, Fig. 183, Pl. I., lying parallel to the quay-line of a harbour, be hauled simultaneously in four directions viz., in the direction, (1), by a force of 120 lbs.; (2), by a force of 100 lbs.; (3), by 200 lbs.; and, (4), by 60 lbs.;—required the resultant pull, R, and the length of the perpendicular distance, p, let fall upon its direction from the centre, O, of the vessel.

$$R = 468 \text{ lbs}; \; p = \tfrac{1}{2} \text{ foot.}$$

7. Under the same conditions, find the turning power, M, due to the four combined pulls, taking a vertical line through the centre of the vessel as the axis of revolution.

$$M = 229 \text{ foot-lbs.}$$

8. In the open cantilever, $A\,B\,E$, Fig. 184, Pl. I., where, $A\,E$, $= 66$ ft., $A\,B = 29$ ft., assume the dead weight, equal to 250 lbs, to be concentrated at, C, defined by, $A\,C = 20$ ft., and an additional load of 200 lbs. to be resting at, D, defined by, $A\,D = 41$ ft., and find the stresses in the rib, $r\,B$; against the abutment at, A; and the tension of anchorage at, B.

Tension, $r\,B = 503$ lbs.
Thrust, $r\,A = 523$ lbs.
Anchorage, $B\,X = 460$ lbs.

9. Place a strut between the points, r and A, and supposing the loads to be proportionately distributed over the joints, find the separate loads acting at, B, r, and E.

$$B = 179{\cdot}3 \text{ lbs.}; \; r = 180{\cdot}3 \text{ lbs.}; \; E = 90{\cdot}4 \text{ lbs.}$$

10. Find the stresses in the bars, $r\,B$, $r\,A$, and the

tension of anchorage, under this new arrangement of the loads.

Tension, $r\,B$ = 430 lbs.
Thrust, $r\,A$ = 212 lbs.
Tension, $B\,X$ = 393 lbs.

11. Find the moment, or turning power, M, of the resultant force about the abutment, A.

$$M = 13185 \text{ foot-lbs.}$$

CHAPTER IV.

CENTRES OF GRAVITY.

1. UNITS OF FORCE.—A unit of force may be distributed throughout a volume such as a cubic foot, or simply over a surface such as a square foot. In this way the term, unit of force, may have two different senses. When treating of *internally* applied force, which may be explained as the *latent force* of the material itself, the first or volume-unit is used, and, when speaking of externally applied force, the second, or surface-unit is taken as the standard of measurement.

2. SPECIFIC GRAVITY.—Specific Gravity is a term defined by the help of a special *volume-unit* of force; as for example by the number of pounds contained in a certain fixed volume of water. Equal volumes of other materials are then compared in weight with water, and tables are formed giving the comparative weights or specific gravities of different substances.

3. CENTRES OF GRAVITY.—The centre of gravity of a body is a point traversed by the resultant of the weights of its separate particles, considered as so many parallel forces.

The centre of gravity of a number of bodies is a point traversed by the resultant of the weights of the separate bodies, looked upon as so many parallel forces, acting through their respective centres of gravity.

If a body be homogeneous; that is, if all its particles have equal specific gravity, and if, moreover, it have a *centre of figure;* that is, a point which bisects every line, drawn centrally through it from end to end or from side to side;—then it is

self-evident that this centre of figure coincides with the centre of gravity of the body.

If a homogeneous body be symmetrically divided by a plane, the centre of gravity of the body will lie somewhere in that plane. The centre of gravity will also be contained in any second plane, which symmetrically divides the body. Hence, it will be found somewhere in the line of intersection of the two dividing planes. If, then, a third symmetrically dividing plane can be determined, it will cut the above line of intersection in a point, marking the centre of gravity of the body.

4. SPECIAL INTEGRATIONS.—In subsequent parts of this work it will sometimes be found necessary to apply forms of integration which it may be well to consider beforehand in a special article; so as to recall to mind the mathematical processes involved in their determination.

Double Integrals, or integrals of the form, $\iint dx\,dy$, take various definite shapes, according to the nature of the surface and the limits of integration involved. When the limits of integration are constant, and the integral is expressed in the form,

$$\int_a^b \int_\alpha^\beta \phi\,(x\,y)\,dx\,dy,$$

signifying that the integration must be made first with respect to, y, between the limits, β and α, considering x, constant;—secondly, that the result of this operation be integrated with respect to, x, between the limits b and a; y, being deemed constant;—it matters little whether the order of integration be permuted, so as to integrate first with respect to, x, secondly with respect to, y.

The case, however, is different when one of the variables, such as, y, has its limits expressed in terms of the other variable, x. The order of integration cannot then be changed without further investigating the nature of the limits. Take the form,

$$\int_a^b \int_{\chi(x)}^{\psi(x)} \phi\,(x\,y)\,dx\,dy.$$

Fig. 94.

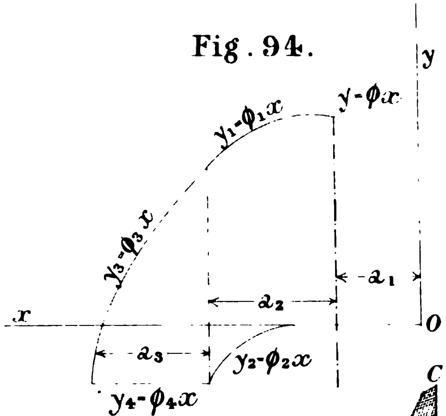


Fig. 95.

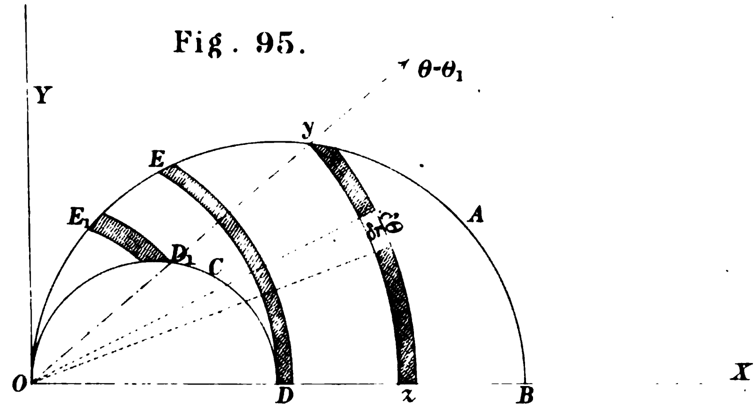


Fig. 96.

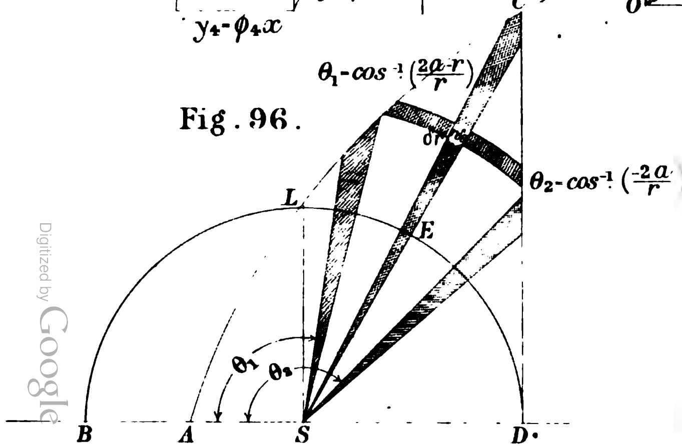


Fig. 97.

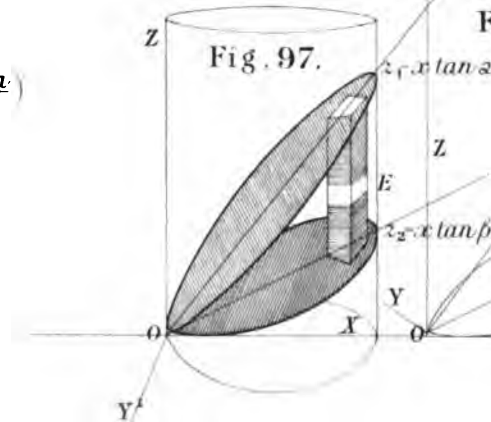
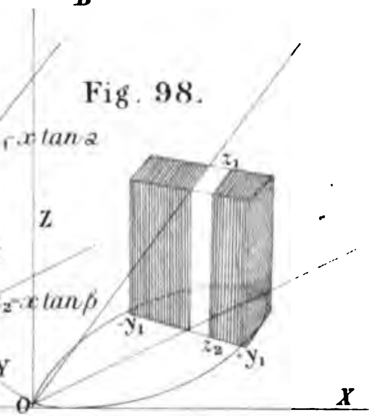


Fig. 98.



Tho.s Kell & Son, Lith.

As a general rule the integration is performed first with respect to that variable having its limits expressed as functions of the other. If the order of integration be reversed, the dependence of the limits, one upon the other, must also be reversed; that is to say, if in one case we integrate first with respect to y, where, y, is a functional form of x;—in the other, when the integration is taken first with respect to, x; x, must be determined as a function of, y.

When the limits are constant, the following forms are equivalent,

$$\int_a^b \int_a^\beta \phi(x).\psi(y)\, dx.\, dy = \int_a^b \phi(x).\, dx. \int_a^\beta \psi(y).\, dy.$$

If an area be bounded by different curves, the integration will generally have to be made part by part. For example, in Fig. 94, the area of a small elemental part of the figure will be represented by, $dx\,dy$; and taking the limits of the whole surface, the general form of the integral will be,

$$\int_x^x \int_{\phi x}^{\phi x} dx\, dy;$$

Owing, however, to the constant changes occurring in the series of curve-limits represented by, $\phi(x)$, it is necessary to take the integration part by part, and sum the results. Thus,

$$\int_o^{a_1} \int_o^{\phi(x)} dx\, dy + \int_{a_1}^{\overline{a_2 + a_1}} \int_{\phi_2(x)}^{\phi_1(x)} dx\, dy +$$

$$\int_{\overline{a_2 + a_1}}^{\overline{a_3 + a^2 + a_1}} \int_{\phi_4(x)}^{\phi_3(x)} dx\, dy.$$

In fact it is imperatively necessary to examine the limits, for, y, expressed as functions of, x, to see if they hold throughout the range of limits, chosen for, x. Should the limits of, y, change their functional form, a line of demarcation must be made where the change takes place, in order to restrict the limits of, x, to that part of the area which corresponds to the particular functional form of, y, considered. It may be gener-

ally stated that every change of curvature, either in the higher or lower limits of, y, imposes a restriction and creates a division. The same rule applies, when the integration is performed first with respect to, x. It is then necessary to observe well that the functional form of limit chosen for, x, obtains throughout the range of limits appertaining to, y.

A good example of this splitting up of the process of integration into several parts is shewn in the operation of finding by polar formulæ the intercepted area, $O\,C\,D\,B\,A\,O$, Fig. 95, contained between the two semi-circles, $O\,A\,B$, and, $O\,C\,D$.

Divide the intercept-area required into a series of zones, $\overline{y\,z}$, bounded by a series of circles described about the centre, O. Let the polar co-ordinates of any elemental area of one of these zones be, r and θ, θ being measured from the initial line, $\overline{O\,X}$. The area of the same element will be, $r.\,\delta\,\theta.\,\delta\,r$. Hence, the area of the whole zone will be

$$\int r.\,\delta r.\,\delta\theta,$$

taken between the limits,

$$\theta = \theta_1 \text{ and } \theta = o.$$

Now,

$$cos.\,\theta_1 = \frac{O\,y}{O\,B} = \frac{r_1}{h};$$

wherefore,

$$\theta_1 = cos.^{-1}.\frac{r_1}{h},$$

in which expression, $O\,B = h$, and r_1 is the radius-vector of the outer circle.

Consequently, the integral giving the area of the zone-strip, $\overline{y\,z}$, can be put into a definite form,

$$\int_{o}^{cos.^{-1}.\frac{r_1}{h}} r\,dr\,\delta\theta,$$

the co-ordinate, θ, having been expressed as a function of the current radius-vector, r_1, of the outer circle.

But it will be seen that these zone-strips can be supposed to extend from the circumference of the outer circle to the axis of, x, only as far as the dividing strip at, $\overline{ED}$. Any strip lying on the near side of, $\overline{ED}$, only reaches from the circumference of the outer to that of the inner circle. Hence, the lower limits of, θ, undergo a change at the line of demarcation, $\overline{ED}$; that is, for values of, r, included between, $\overline{OB}$, and $\overline{OD}$, the limits are as above stated; viz. $\cos.^{-1}\frac{r_1}{h}$. and zero; but for values of, r, included between $\overline{OD}$ and the origin, the limits of, θ, change to

$$\cos.^{-1}\frac{r_1}{h}. \text{ and } \cos.^{-1}\frac{r_2}{c},$$

where, r_2, is the current radius-vector of the inner circle, and c, equal the diameter, $\overline{OD}$, of the same circle.

Taking the sum of the two parts of the integration, we find the area of the whole intercept equal to

$$\int_c^h\int_o^{\cos.^{-1}r} r\,dr\,d\theta + \int_o^c\int_{\cos.^{-1}\frac{r}{c}}^{\cos.^{-1}\frac{r}{h}} r\,dr\,d\theta;$$

the particular forms of, r, viz. r_1 and r_2, being merged in the general symbol for the radius-vector; since their limits are adequately defined by the constants, h, c, and c, o, accompanying the second sign of integration.

Let an arc of a parabola, ALC, Fig. 96, meet a semicircle, BLD, in a point, L, such that the part, $\overline{LB}$, cut off shall subtend a right angle at, S. Under these conditions, it is required to find the area of the intercept, LDC, included between the arcs of the circle and parabola, and the vertical line, $\overline{DC}$. Let, S, be the centre of the circle as well as the focus of the parabola; r, the radius of the circle, equal $2a$, and let angles, θ, be measured from the initial line, $\overline{SB}$.

The equation to the line, $\overline{DC}$, will be

$$r \cos. \theta = -2a,$$

since for that line the angle, θ, is always greater than 90°.

The equation to the circle will be

$$r = 2a,$$

and the equation of the parabola,

$$r = \frac{a}{\cos.^2 \dfrac{\theta}{2}}.$$

The line, $\overline{SC}$, is a radius-vector common to the line, $\overline{DC}$, and the parabolic arc ;—wherefore its value must satisfy both these lines, *i.e.*,

$$\overline{SC} = \frac{a}{\cos.^2 \dfrac{\theta}{2}} = \frac{2a}{1 + \cos. \theta}$$

and

$$\overline{SC} = \frac{-2a}{\cos. \theta};$$

whence, by substitution for, $\cos. \theta$,

$$\overline{SC} = 4a.$$

The general integral expressing the area of the intercept is

$$\iint r \, d\theta \, dr.$$

Integrating first with respect to, r, it will be seen that the radius-vector, $\overline{SC}$, determines a line of demarcation and a division of the process of integration into two parts; because, from $\overline{SL}$ to $\overline{SC}$, the intercept, LEC, is bounded by the arcs of the parabola and circle; whereas, from $\overline{SC}$ to $\overline{SD}$, the part-area, CED, is limited by the straight line, $\overline{DC}$, and the circular arc. Hence, the limits of, r, expressed as functional forms of θ, undergo a change in nature at the line of division, $\overline{SC}$., so that, r, must be taken between different limits in the first to what it is in the second part of the integration.

The line $\overline{SC}$, corresponds to an angle, θ, determined by the relation

$$\cos. \theta = -\frac{SD}{SC} = -\frac{2a}{4a} = -\frac{1}{2};$$

consequently,

$$\theta = \frac{2}{3} . \pi$$

The complete integral, giving the entire area of the intercept, will be made up of the two parts

$$\int_{\frac{2}{3}\pi}^{\frac{\pi}{2}} \int_{2a}^{a\, sec.^2 \frac{\theta}{2}} r\, d\theta\, dr + \int_{\pi}^{\frac{2}{3}\pi} \int_{2a}^{-2a\, sec.\, \theta} r\, d\theta\, dr$$

If the integration were performed first with respect to θ, the area required could be found in one operation, symbolically expressed by

$$\int_{2a}^{4a} \int_{\theta_2}^{\theta_1} r\, dr\, d\theta\,;$$

in which the terms, θ_1 and θ_2, represent the limits of θ, determined as functional forms of, r; so that

$$\theta_1, = \cos.^{-1} \frac{2a - r}{r}$$

$$\theta_2, = \cos.^{-1} \left(-\frac{2a}{r} \right)$$

Let the equation to the right circular cylinder given in Fig. 97, be $x^2 + y^2 - 2ax = o$, and let two planes passing through the origin perpendicularly to the plane, $\overline{ZX}$, cut off from the cylinder a wedge-shaped slice of its volume, as shewn in the figure.

Suppose the equations to the traces of the two dividing planes on the plane of, $\overline{ZX}$, to be,

$$z_1 = x\, tan.\, a$$

$$z_2 = x\, tan.\, \beta$$

A small elemental volume of this wedge-shaped slice will be equal to

$$dx.\, dy.\, dz.,$$

and the area of the whole volume cut-off can be put into the general form,

$$\int\int\int dx.\,dy.\,dz.$$

Integrating first with respect to, z, it is necessary to extend the elemental area, $E = dx.\ dy.\ dz.$, up and down, between the limits, $z_1 = x\ tan.\ \alpha$, and $z_2 = tan.\ \beta$, forming in this way a small elemental column such as that drawn and shaded in Fig. 97.

Integrating next with respect to, y, the elemental column, $\overline{z_1 z_2}$, must be further developed, so as to reach from one side of the cylindrical surface to the other [Fig. 98]. This can be effected by taking the limits of y, from y_1, to $-y_1$, where by the equation to the cylinder

$$y_1 = \sqrt{2a\,x - x^2}$$
$$-y_1 = -\sqrt{2a\,x - x^2}$$

Lastly, the prismatic volume resulting from the last integration must be taken between the limits, $x = 2\,a$, and $x = o$.

The complete integral takes the form

$$\int_o^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}}\int_{x\ tan.\ \beta}^{x\ tan.\ \alpha} dx.\,dy.\,dz =$$

$$\int_o^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} x\,[tan.\ \alpha - tan.\ \beta]\,dx.\,dy. =$$

$$2.\int_o^{2a} [tan.\ \alpha - tan.\ \beta]\,x.\,\sqrt{2a\,x - x^2.\,dx}.$$

But,

$$\int x\,\sqrt{2a\,x - x.^2\,dx.} = \int x^{\frac{3}{2}}\,\sqrt{2a - x.\,dx}.$$

$$\text{Let, } \frac{dv}{dx} = \sqrt{2a - x}\,;\ \text{then, } v = -\frac{2}{3}\,[2a - x]^{\frac{3}{2}}.$$

By the usual formula,

$$\int x^{\frac{3}{2}}\,\sqrt{2a - x}.\,dx = \int u.\,dv = u.\,v - \int v.dv.$$

Hence,

$$\int x^{\frac{3}{2}}\sqrt{2a-x}.\,dx = -x.^{\frac{3}{2}}\frac{2}{3}[2a-x]^{\frac{3}{2}}$$

$$+\int\frac{2}{3}[2a-x]^{\frac{3}{2}}.\ \frac{3}{2}x^{\frac{1}{2}}\,dx.$$

$$= -\frac{2}{3}[2a\,x - x^2]^{\frac{3}{2}} + \int[2a-x]^{\frac{3}{2}}_{.}\,x^{\frac{1}{2}}\,dx.$$

Again,

$$\int[2a-x].^{\frac{3}{2}}x^{\frac{1}{2}}.\,dx.$$

$$=\int(2a-x)\,(2a-x).^{\frac{1}{2}}x^{\frac{1}{2}}.\,dx$$

$$=\int 2a\,(2a-x).^{\frac{1}{2}}x^{\frac{1}{2}}.\,dx - \int x.^{\frac{3}{2}}[2a-x]^{\frac{1}{2}}\,dx.$$

By substitution and transposition,

$$\int x\sqrt{2a\,x - x^2}.\,dx = -\frac{1}{3}[2a\,x-x^2]^{\frac{3}{2}} + a\int[2a\,x-x^2].^{\frac{1}{2}}dx.$$

The first term of the right-hand member of this equation vanishes for both the limits, $2\,a$ and o;—the second term being integrated gives,

$$a\int_o^{2a}[2a\,x-x^2]^{\frac{1}{2}}_{.}\,dx = a\int_o^{2a}[a^2-(x-a)^2]^{\frac{1}{2}}_{.}\,d\,(x-a)$$

Let, $z=(x-a)$; then the integral takes the form,

$$a\int_{-a}^{a}[a^2-z^2].^{\frac{1}{2}}\,dz.$$

Let, $(a^2-z^2)^{\frac{1}{2}} = u$; $d\,z = dv$; then

$$a\int_{-a}^{a}u\,dv = a\left[uv - \int v\,du\right]$$

$$= a\left[(a^2-z^2).^{\frac{1}{2}}z + \int\frac{z^2}{\sqrt{a^2-z^2}}.\,dz\right]^{+a}_{-a}$$

$$= a\left[(a^2-z^2)^{\frac{1}{2}}.z - \int\frac{a^2-z^2}{\sqrt{a^2-z^2}}.\,dz + \int\frac{a^2}{\sqrt{a^2-z^2}}.\,dz\right]^{+a}_{-a}$$

By transposition,

$$2.\, a\int_{-a}^{a} \sqrt{a^2 - z^2}.\, dz = a\left[(a^2 - z^2)^{\frac{1}{2}}.\, z + \right.$$

$$\left. + \int \frac{a^2}{\sqrt{a^2 - z^2}}.\, dz\right]_{-a}^{+a}$$

But,

$$\int \frac{dz}{\sqrt{a^2 - z^2}} = sin.^{-1}\frac{z}{a};$$

Whence,

$$a\int_{-a}^{a} \sqrt{a^2 - z^2}\, dz = \frac{a}{2}\left[(a^2 - z^2)^{\frac{1}{2}}_{.}\, z + a^2\, sin.^{-1}\frac{z}{a}\right]_{-a}^{+a}$$

This expression taken between the limits, $z = a$, and $z = -a$, gives, for the definite value of the above integral,

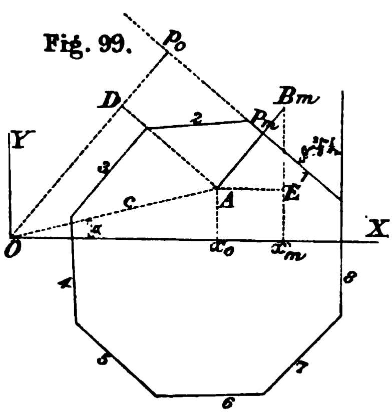
$$a\int_{-a}^{a} \sqrt{a^2 - z^2}.\, dz = \frac{\pi}{2}.\, a^3.$$

$$= \int_{0}^{2a} x.\, \sqrt{2a\, x - x^2}.\, dx$$

The final value found for the wedge-shaped slice is therefore,

$$\pi\, a^3\, [tan.\, \alpha - tan.\, \beta].$$

Fig. 99.



5. Examples of Centres of Gravity.—From any point, A, Fig. 99, let perpendiculars be drawn upon the sides, and pro-

duced in a certain ratio, $\frac{A\,B_m}{A\,p_m} = r$, beyond the sides of an equiangular polygon, 1, 2, 3, 4, . . 7, 8. Let a series of equally heavy particles be placed at the points, B, and let it be required to find the centre of gravity of the system of particles so distributed.

Take the axis of, x, parallel to any one of the perpendiculars drawn from, A, upon the sides, as for example to that drawn perpendicularly to the side, 8. Let the radius-vector, $\overline{O\,A}$, of the point, A, equal, c; and let the angle, $A\hat{O}X = a$.

If the polygon have, n, sides, the supplement of each of its angles must be equal to $\beta = \frac{2\,\pi.}{n}$ Let the line-perpendicular, Ap_m, be represented by, p_m; then

$$\begin{aligned} p_m = Op_o - \overline{O\,D} &= Op_o - \overline{O\,A}\ cos.\ A\hat{O}D \\ &= Op_o - \overline{O\,A}.\,cos.\ [p_o\hat{O}x - a] \\ &= p_o - c.\ cos.\ [m\,\beta - a]: \end{aligned}$$

in which equation, m, equals the number of sides of the polygon included between the axis of, x, and the side considered. For the arrangement given in the Figure, $m = 1$

Let, x_m, be the abscissa of the point, B_m;

$$\begin{aligned} x_m &= Ox_o + \overline{x_o\,x_m} \\ &= c.\ cos.\ a + \overline{A\ E} = c.\ cos.\ a + \overline{B_m\ A}.\ cos.\ B_m\hat{A}E; \\ &\qquad\qquad = c.\ cos.\ a + r.\ Ap_m.\ cos.\ B_m\hat{A}E; \\ &= c.\ cos.\ a + r\,[p_o - c.\ cos.\ (m\,\beta - a)].\ cos.\ m\,\beta. \end{aligned}$$

Similarly,

$$y_m = c.\ sin.\ a + r\,[p_o - c.\ cos.\ (m\,\beta - a)].\ sin.\ m\,\beta.$$

Represent the weight of each particle by unity, and let the abscissa of the centre of gravity of the system be denominated, $\overline{x}$;—in that case, since there are, n, distributed particles,

$$n.\,\overline{x} = \Sigma_0^{n-1}\,[r\,\{p_o - c.\,cos.\,(m\,\beta - a)\}\ cos.\ m\,\beta + c.\,cos.\,a].\ unity;$$

which expression can be put into the form of three separate sums, namely,

$$n.\bar{x} = \Sigma_0^{n-1}.\, r.\, p_o\, cos.\, m\,\beta - \Sigma_0^{n-1}\, r.\, c.\, cos.\, (m\,\beta - a)\, cos.\, m\,\beta + \Sigma_0^{n-1}.\, c.\, cos.\, a.$$

The second term of the right-hand member of this equation comprises the sum of a series of cosines in arithmetical progression; viz.,

$$\Sigma_0^{n-1}\, cos.\, (m\,\beta - a)\, cos.\, m\,\beta = \tfrac{1}{2}\,\Sigma_0^{n-1}.\, [cos.\, (2\,m\,\beta - a) + cos.\, a].$$

The summation of a series of this type will consist of two parts. First, there will be the series,

$$cos.\, (-a) + cos.\, (2\,\beta - a) + cos.\, (4\,\beta - a) + \ldots\ldots\ldots + cos.\, (2\,(n-1)\,\beta - a),$$

in which the angles increase by the arithmetical difference, $2\,\beta$. According to the ordinary formula the sum of the above series can be expressed as,

$$\frac{cos.\, [(n-1)\,\beta - a]\, sin.\, n\,\beta.}{sin.\,\beta}$$

But, since, in this case, $\beta = \dfrac{2\,\pi}{n}$; $n\,\beta = 2\,\pi$, and $sin.\, n\,\beta = o$. This series, therefore, vanishes, and there remains only the second part of the sum, viz.,

$$\tfrac{1}{2}\,\Sigma_0^{n-1}\, cos.\, a.$$

Substituting this value of, $\Sigma_0^{n-1}\, cos.\, (m\,\beta - a)\, cos.\, m\,\beta$, in the value of $n.\,\bar{x}$, we find

$$n.\,\bar{x} = r.\,\Sigma_0^{n-1}\, p_o\, cos.\, m\,\beta - \tfrac{1}{2}\, r\, c\,\Sigma_0^{n-1}\, cos.\, a + \Sigma_0^{n-1}\, c.\, cos.\, a.$$

Hence, since the symbol, $\Sigma_0^{n-1}\, cos.\, a$, implies the repetition of the same quantity, $cos.\, a$, n times; it follows that

$$\bar{x} = \frac{r.}{n}\,\Sigma_0^{n-1}\, p_o\, cos.\, m\,\beta - \tfrac{1}{2}\, r.\, c.\, cos.\, a + c.\, cos.\, a,$$

and similarly that

$$\overline{y} = \frac{r.}{n} \Sigma^{n-1} p_0 \, sin. \, m \, \beta - \tfrac{1}{2} \, r \, c. \, sin. \, a + c. \, sin. \, a.$$

II. 2° *Example.*—Suppose it were required to find the area comprised within the limits, $E F E_1 F_1$, Fig. 100.

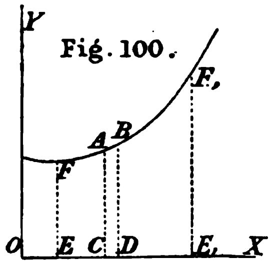


Fig. 100.

Let the abscissa of a very small elemental strip, $A\, D$, be x. The area of the strip will ultimately be equal to

$$\overline{C D}. \tfrac{1}{2} [\overline{A C} + \overline{B D}];$$

that is to

$$\Delta x. \tfrac{1}{2} [y + \overline{y + \Delta y}].$$

But, in the limit, as the elemental strip is gradually narrowed, the value of, Δy, becomes indefinitely small, and may, therefore, be neglected. In that case,

$$\text{Area, } A\, B\, C\, D, = y. \, dx.$$

The distance of the centre of gravity of the strip from the axis of, y, is very approximately equal to, x. Hence the moment of the same elemental area about an axis perpendicular to the plane of $\overline{x\, y}$ and passing through the point, O, is

$$y. \, dx. \, x,$$

and the moment of the whole figure about the same axis,

$$M_1 = \int x \, y. \, dx.$$

Let, $\bar{x}$, equal the distance of the centre of gravity of the materialised area from the axis of y; then by equality of moments

$$M_1 = \bar{x} . \int y\, dx,$$

or,

$$\bar{x} = \frac{M_1}{\int y\, dx} = \frac{\int x . y\, dx}{\int . y . \, dx};$$

in which, $\int y\ dx$, represents the area of the surface considered.

Similarly, the moment of the strip, $A\ D$, about an axis through, O, found by supposing the forces of gravity to be turned through a right angle, so as to act in lines parallel to the axis of x, will be represented by

$$y . \, dx . \frac{y}{2};$$

since, in the limit, $\frac{y}{2}$, is the ordinate of the centre of gravity of the strip. The corresponding moment of the entire surface will be equal to

$$M_2 = \tfrac{1}{2} \int y^2 . \, dx,$$

and the ordinate of its centre of gravity,

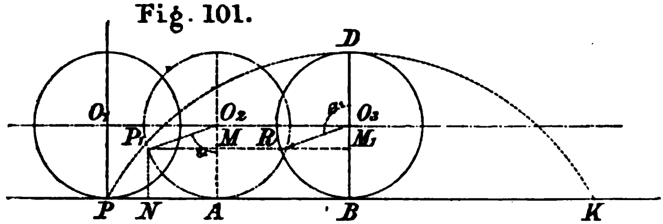
$$\bar{y} = \frac{\tfrac{1}{2} \int y .^2\, dx}{\int y . \, dx},$$

which, when, y, has been expressed as a function of, x, must be taken between the same limits for, x, as in the first integral, used to determine, $\bar{x}$.

III. 3° *Example.*—Let a point, P, in a circle rolling along a straight line, $\overline{PK}$, Fig. 101, trace out the cycloidal path,

$P D K$. After a definite lapse of time from the commencement of motion, let the position of, P, on the path it describes, be represented by, P_1; and let, A, be the corresponding point on the directing-line, $\overline{P K}$.

Fig. 101.



Take, P, as the origin of movement and co-ordinates, and let,

$$\overline{PN} = x; \; \overline{NP_1} = y;$$

then

$$\begin{aligned} x &= \overline{PN} = \overline{PA} - \overline{NA} = \overline{P_1A} - \overline{NA} = \\ &= \overline{O_2P_1}.\,\theta - \overline{O_2P_1}.\,sin.\,\theta \\ &= a.\,\theta - a.\,sin.\,\theta = a\,[\theta - sin.\,\theta], \end{aligned}$$

and

$$\begin{aligned} y &= \overline{NP_1} = \overline{AM} = \overline{O_2A} - \overline{O_2M} = \\ &= a - a\,cos.\,\theta = a\,[1 - cos.\,\theta] \\ &= a\,vers.\,\theta \end{aligned}$$

If the origin be taken at, D, instead of at, P, we shall have for the values of x_1, y_1, the co-ordinates of any point in the cycloidal curve,

$$y_1 = \overline{O_2O_3} + \overline{MP_1} = \overline{AB} + \overline{MP_1} = \overline{PB - PA} + \overline{MP_1}$$

But, $\overline{PB}$ = the semicircle, $\widehat{DRB}$; $PA = P_1A$ = arc $\widehat{RB}$;—therefore,

$$\begin{aligned} y_1 &= \widehat{DRB} - \widehat{RB} + \overline{MP_1} \\ &= \widehat{DR} + \overline{MP_1} = \widehat{DR} + \overline{M_1R} = a.\,\theta_1 + a\,sin.\,\theta_1 = \\ &\qquad a\,[\theta_1 + sin.\,\theta_1], \end{aligned}$$

and,

$$x_1 = \overline{DM_1} = \overline{DO_3} + \overline{O_3M_1}$$
$$= a - a\ cos.\ \theta_1 = a\ vers.\ \theta_1.$$

Whence,

$$vers.\ \theta_1 = \frac{x_1}{a};\ \ and\ \ \theta_1 = vers.^{-1}\frac{x_1}{a}$$

$$sin.\ \theta_1 = \sqrt{1 - cos.^2\theta_1} = \sqrt{1 - (1 - vers.\ \theta_1)^2}$$
$$= \frac{1}{a}\sqrt{2\,a\,x_1 - x_1^2}$$

or,

$$a\,sin.\ \theta_1 = \sqrt{2\,a\,x_1 - x_1^2}$$

Consequently,

$$y_1 = a\,\theta_1 + a\ sin.\ \theta_1$$
$$= a.\ vers.^{-1}\frac{x_1}{a} + \sqrt{2ax_1 - x_1^2}$$

and,

$$\frac{dy_1}{dx_1} = \frac{a}{\sqrt{2ax_1 - x_1^2}} + \frac{2a - 2x_1}{2\sqrt{2ax_1 - x_1^2}}$$
$$= \frac{2a - x_1}{\sqrt{2ax_1 - x_1^2}}$$
$$= \frac{(2a - x_1)\sqrt{2ax_1 - x_1^2}}{(2a - x_1)\,x_1} = \frac{\sqrt{2ax_1 - x_1^2}}{x_1}.$$

If it were required to find the centre of gravity of the part area, DBP; the co-ordinates of that centre, relatively to the origin, D, would be furnished by the usual formula,

$$x_0 = \frac{\int_0^{2a} x\,y\,dx}{\int_0^{2a} y\,.\,dx};\ \ y_0 = \frac{\frac{1}{2}\int_0^{2a} y^2\,dx}{\int_0^{2a} y\,dx}$$

Now,

$$\int xydx = \frac{x^2}{2}.\, y - \int \frac{x^2}{2}.\frac{dy}{dx}.\, dx$$

$$= y.\frac{x^2}{2} - \int \frac{x^2}{2}.\, \frac{\sqrt{2ax - x^2}}{x}.\, dx$$

$$= y.\frac{x^2}{2} - \int \frac{x}{2}.\, \sqrt{2ax - x^2}.\, dx$$

But (*see* Special Integrations, p. 148),

$$\int_0^{2a} x.\, \sqrt{2ax - x^2}.\, dx = \frac{\pi\, a^3}{2}$$

Hence, since when $x = 0$; $y = 0$; when $x = 2a$; $y = \pi a$,

$$\int_0^{2a} x.y.dx = \frac{(2a)^2.\, \pi a}{2} - \frac{\pi a^3}{4}$$

$$= \frac{7}{4}.\, \pi a^3.$$

Again,

$$\int y.\, dx = y.\, x - \int x.\, \frac{dy}{dx}.\, dx$$

$$= y.\, x - \int \sqrt{2ax - x^2}.\, dx$$

But (*see* Integration, p. 147),

$$\int_0^{2a} \sqrt{2ax - x^2}.\, dx = \frac{\pi a^2}{2}$$

Therefore,

$$\int_0^{2a} ydx = 2\pi.a^2 - \frac{\pi a^2}{2}$$

$$= \frac{3}{2}\pi.a^2$$

Finally,

$$x_o = \frac{\frac{7}{4}\,.\,\pi a^3}{\frac{3}{2}\,.\,\pi a^2} = \frac{7}{6}a.$$

Further,

$$\int y^2\,dx = y^2.x - \int x\,.\,2y\,.\,\frac{dy}{dx}\,.\,dx$$

$$= y^2x - 2\int y\,\sqrt{2ax - x^2}\,.\,dx$$

$$= y^2x - 2\int [2ax - x^2]\,dx$$

$$- 2a\int \sqrt{2ax - x^2}\,.\,vers.^{-1}\frac{x}{a}\,dx$$

$$= y^2x - \frac{4ax^2}{2} + \frac{2x^3}{3} - 2a\int \sqrt{2ax - x^2}\,.\,vers.^{-1}\frac{x}{a}\,dx$$

Wherefore,

$$\int_o^{2a} y^2dx = 2a.\,(\pi a)^2 - \frac{8}{3}\,.\,a^3 - 2a\int_o^{2a} \sqrt{2ax - x^2}\,.\,vers.^{-1}\frac{x}{a}\,.dx.$$

To find, $\int_o^{2a} \sqrt{2a\,x - x^2}.\,vers.^{-1}\frac{x}{a}.\,dx.$; assume, $vers.^{-1}\frac{x}{a} = \theta$,

from which by differentiation we have,

$$d\,\theta.\,\sqrt{2a\,x - x^2} = dx\,;$$

also,

$$2ax = 2a^2\,vers.\,\theta$$

$$x^2 = a^2.\,vers.^2\theta.$$

Substituting the above values, the integral required takes the form,

$$\int a^2.\,sin.^2\,\theta.\,\theta.\,d\theta,$$

and the new limits for, θ, are π and o.

Let, $sin.^2 \theta = d v$; then

$$v = \int sin.^2 \theta \, d\theta =$$

$$= \int sin. \, \theta. \, sin. \, \theta. \, d\theta$$

$$= - sin. \, \theta. \, cos. \, \theta + \int cos.^2 \theta \, d\theta$$

$$= - sin. \, \theta. \, cos. \, \theta + \int [1 - sin.^2 \theta] \, d\theta;$$

whence, by transposition

$$\int sin.^2 \theta \, d\theta = v = \frac{\theta}{2} - \frac{sin. \, \theta \, cos. \, \theta}{2};$$

and,

$$\int \theta. \, sin.^2 \theta. \, d\theta = \int u. \, dv = uv - \int v \, du$$

$$= \frac{\theta^2}{2} - \frac{\theta. \, sin. \, \theta \, cos. \, \theta}{2}$$

$$- \int \left[\frac{\theta}{2} - \frac{sin. \, \theta \, cos. \, \theta}{2} \right] d\theta.$$

$$= \frac{\theta^2}{2} - \frac{\theta \, sin. \, \theta \, cos. \, \theta}{2} - \frac{\theta^2}{4} + \frac{sin.^2 \theta}{4}.$$

This result being taken between the limits, $\theta = \pi$ and $\theta = o$, gives the value, $\frac{\pi^2}{4}$; hence

$$a^2 \int_o^{\pi} sin.^2 \theta. \, \theta \, d\theta = \int_o^{2a} \sqrt{2ax - x^2}. \, vers.^{-1} \frac{x}{a}. \, dx = \frac{(\pi \, a)^2}{4}.$$

Substituting this term in the general equation, we obtain

$$\int_o^{2a} y^2. \, dx = 2a. \, (\pi \, a)^2 - \frac{8}{3}. \, a^3 - 2a. \, \frac{(\pi \, a)^2}{4}$$

$$= \frac{3}{2} \pi^2 a^3 - \frac{8}{3} a^3.$$

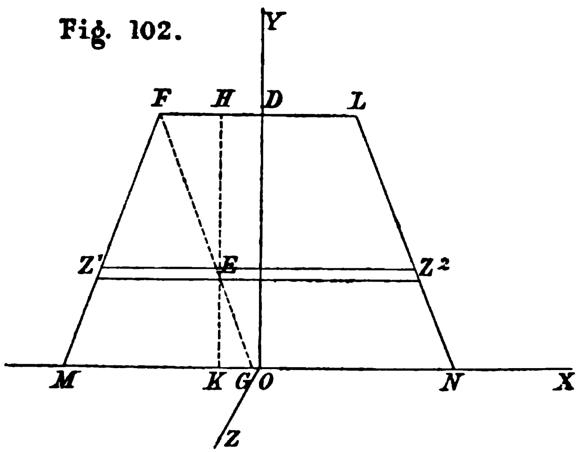
Finally,

$$\bar{y}_0 = \frac{\frac{1}{2}\int_0^{2a} y^2\,dx}{\int_0^{2a} y\,dx} = \frac{\frac{1}{2}\,a^3\left[\frac{3}{2}\pi^2 - \frac{8}{3}\right]}{\frac{3}{2}.\,\pi\,a^2}.$$

$$= \frac{a}{3\,\pi}.\left[\frac{3}{2}\,\pi^2 - \frac{8}{3}\right]$$

IV. 4° *Example*:—Let it be required to find the centre of gravity of the trapezoidal figure, $F\,L\,M\,N$, Fig. 102.

Fig. 102.



It is evident that the centre of gravity must lie somewhere in the line, $\overline{O\,D}$, which divides the whole figure into two equal and symmetrical parts; since $\overline{O\,D}$ is the locus of the centres of all lines drawn parallel to, $\overline{F\,L}$ or $\overline{M\,N}$.

Take the line, $\overline{O\,D}$, as the axis of, y, and the line, $\overline{O\,N}$, as that of x, the point, O, being taken as the origin of co-ordinates.

The elemental area of any strip, $\overline{Z_1\,Z_2}$, will be ultimately equal to

$$Z_1\,Z_2.\,d\,y.\;sin.\;N\,\hat{O}\,D.$$

Let, y, be the ordinate of the strip, $\overline{Z_1 Z_2}$;—then the moment of the strip relatively to an axis, $\overline{OZ}$, perpendicular to the plane of the trapezoid, will be

$$\overline{Z_1 Z_2}.\ y.\ dy.\ \sin.\ N\hat{O}D,$$

and the sum of all such moments contained within the limits, $y = \overline{OD}$ and $y = o$,

$$\int_o^{\overline{OD}} \overline{Z_1 Z_2}.\ y.\ dy\ \sin.\ N\hat{O}D,$$

which, being divided by the total area of the trapezoid, will give the value of the ordinate of the centre of gravity equal to

$$\bar{y} = \frac{\int_o^{\overline{OD}} \overline{Z_1 Z_2}.\ y.\ dy.\ \sin.\ N\hat{O}D}{\int_o^{\overline{OD}} \overline{Z_1 Z_2}.\ dy.\ \sin.\ N\hat{O}D}.$$

Now,

$$\overline{Z_1 Z_2} = \overline{Z_2 E} + \overline{E Z_1} = \overline{FL} + \overline{E Z_1} = b + \overline{E Z_1}$$

and,

$$\frac{E Z_1}{GM} = \frac{FE}{FG} = \frac{HE}{HK} = \frac{OD - y}{OD}. = \left[1 - \frac{y}{OD}\right]$$

also,

$$\overline{GM} = \overline{MN} - \overline{NG} = B - b;$$

so that,

$$\overline{E Z_1} = (B - b)\left[1 - \frac{y}{OD}.\right]$$

Wherefore,

$$\bar{y} = \frac{\int_0^{\overline{OD}} b.\,y\,dy\,sin.\,N\hat{O}D + \int_0^{\overline{OD}} (B-b)\left[1-\frac{y}{OD}\right]y.\,dy\,sin.\,N\hat{O}D}{\int_0^{\overline{OD}} b.\,dy\,sin.\,N\hat{O}D + \int_0^{\overline{OD}} (B-b)\left[1-\frac{y}{OD}\right]dy.\,sin.\,N\hat{O}D}$$

but,

$$\int_0^{\overline{OD}} b\,y\,dy\,sin.\,N\hat{O}D = b.\,\frac{\overline{OD}^2}{2}.\,sin.\,N\hat{O}D,$$

and,

$$\int_0^{\overline{OD}} (B-b)\left[1-\frac{y}{OD}\right].\,y\,dy\,sin.\,N\hat{O}D =$$

$$= (B-b)\left[\frac{OD^2}{2} - \frac{OD^2}{3}\right] sin.\,N\hat{O}D.$$

Summing these two results, we find for the value of the numerator in the fractional form of, $\bar{y}$,

$$\frac{OD^2}{2}.\,sin.\,N\hat{O}D.\left[B - \frac{2}{3}(B-b)\right].$$

Again,

$$\int_0^{\overline{OD}} b.\,dy.\,sin.\,N\hat{O}D = b.\,\overline{OD}.\,sin.\,N\hat{O}D;$$

and

$$\int_0^{\overline{OD}} (B-b)\left[1-\frac{y}{OD}\right]dy.\,sin.\,N\hat{O}D =$$

$$= (B-b)\,\overline{OD}\,sin.\,N\hat{O}D - (B-b)\,\frac{OD}{2}.\,sin.\,N\hat{O}D.$$

Adding these two results we find the denominator of the fraction equal to

$$\frac{\overline{OD}}{2} . \sin. N\hat{O}D. [B + b] ;$$

and dividing the numerator already found by this denominator,

$$\bar{y} = \frac{\frac{\overline{OD}^2}{2} . \sin. N\hat{O}D \left[B - \frac{2}{3} (B - b) \right]}{\frac{\overline{OD}}{2} . \sin. N\hat{O}D [B + b]}$$

$$= \frac{\overline{OD} \left[B - \frac{2}{3} (B - b) \right]}{B + b}$$

$$= \frac{\frac{\overline{OD}}{2} \left[\overline{B + b} - \frac{1}{3} (B - b) \right]}{B + b}$$

$$= \frac{OD}{2} \left[1 - \frac{1}{3} . \frac{B - b}{B + b} \right] .$$

V. 5° *Example*:—Let it be required to find the centre of gravity of the circular sector, OAC, Fig. 103.

Represent the angle, $X\hat{O}A$, by the symbol, θ; and let, r, equal the radius of the given circle.

The area of an element of any strip, $\overline{BD}$, will be, $dx. dy$; and, therefore, the area of the whole strip will be, $\int dx\, dy$, taken within limits, B and D.

The limit, B, is the general limit expressed by the equation of the given circle; namely,

$$x = \sqrt{r^2 - y^2},$$

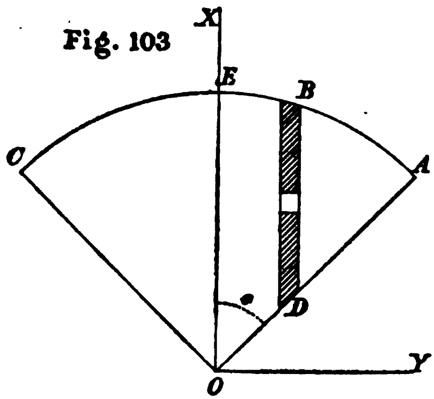
and the limit, D, is the general limit, expressed by the equation to the line, $\overline{AO}$; that is

$$x = y \cot. \theta ;$$

So that the area of the strip can be put into the form,

$$\int_{y \cot.\theta}^{\sqrt{r^2-y^2}} dy.dx = [\sqrt{r^2-y^2} - y \cot.\theta]\, dy$$

Fig. 103



Summing the whole of similar strips contained between the limits, $y = r \sin.\theta$ and $y = o$, we obtain the area of the half-sector, AOE, which will be

$$\int_0^{r \sin.\theta} [\sqrt{r^2-y^2} - y \cot.\theta]\, dy =$$

$$= \frac{y\sqrt{r^2-y^2}}{2} + \frac{r^2}{2} \sin.^{-1} \frac{y}{r} - \frac{y^2}{2} \cot.\theta,$$

taken between the given limits; or

$$\int_0^{r \sin.\theta} [\sqrt{r^2-y^2} - y \cot.\theta]\, dy =$$

$$= \frac{r \sin.\theta \sqrt{r^2 - r^2 \sin.^2\theta}}{2} + \frac{r^2\theta}{2} - \frac{r^2 \sin.\theta \cos.\theta}{2} =$$

$$= \frac{r^2 \sin.\theta.\cos.\theta}{2} + \frac{r^2\theta}{2} - \frac{r^2 \sin.\theta.\cos.\theta}{2} =$$

$$= \frac{r^2.\theta}{2}.$$

Hence, the area of the whole sector, AOC, will be equal to, $r^2.\theta$. Secondly, the moment of the element, $dx\,dy$, relatively to an axis traversing, O, at right angles to the plane, YX, can be expressed by

$$\iint x\,dx\,dy,$$

and the moment of the half sector, relatively to the same axis, by

$$\int_0^{r\,sin.\,\theta}\int_{y\,cot.\,\theta}^{\sqrt{r^2-y^2}} x\,dy.\,dx =$$

$$= \int_0^{r\,sin.\,\theta}\frac{1}{2}[r^2-y^2]\,dy - \int_0^{r\,sin.\,\theta}\frac{1}{2}y^2\,cot.^2\,\theta\,dy,$$

$$= \frac{r^3.\,sin.\,\theta}{2} - \frac{r^3\,sin.^3\,\theta}{6} - \frac{r^3\,sin.^3\,\theta}{6}\,cot.^2\,\theta, =$$

$$= \frac{1}{3}\,r^3\,sin.\,\theta.$$

Hence the moment of the whole sector will be equal to, $\frac{2}{3}\,r^3\,sin.\,\theta$; and the ordinate of the centre of gravity,

$$\overline{x} = \frac{2}{3}\cdot\frac{r^3\,sin.\,\theta}{r^2\theta} = \frac{2}{3}\cdot\frac{r\,sin.\,\theta}{\theta}.$$

VI. 6° *Example*:—The centre of gravity of a semicircular wedge, Fig. 104, is found by taking the area of an elemental section, $\overline{A\,B}$, parallel to the plane, $Z\,Y$, which is equal to

$$\overline{A\,B} \times z = 2\,y \times z$$

By the equation of the circle,

$$y = \sqrt{r^2 - x^2},$$

and by similar triangles,

$$\frac{z}{z_1} = \frac{x}{x_1}.$$

Hence, the sum of the areas of all such elemental sections, $\overline{A\,B}$; or, in other words, the volume of the circular wedge, may be represented by

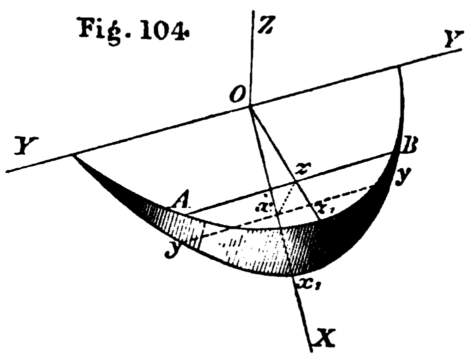
$$2\int_0^r \frac{z_1}{x_1} \,.\, x \,.\, \sqrt{r^2 - x^2} \,.\, dx = 2\,\frac{z_1}{x_1}\int_0^r x \,.\, \sqrt{r^2 - x^2} \,.\, dx =$$

$$= -\,2\,\frac{z_1}{x_1}\left[\frac{1}{3}(r^2 - x^2)^{\frac{3}{2}}\right]$$

taken between the given limits, or volume,

$$= \frac{2}{3}\,z^1 \,.\, r^2$$

Fig. 104.



Again the moment of the elemental section, $\overline{A\,B}$, relatively to an axis in the plane, $\overline{Y\,Z}$, will be

$$\left[2 \,.\, \frac{z_1}{x_1} \,.\, x\,\sqrt{r^2 - x^2}\right] .\, x,$$

and the sum of all such moments,

$$2\,\frac{z^1}{x_1}\int_0^r x^2\,\sqrt{r^2 - x^2} \,.\, dx = 2\,\frac{z_1}{x_1}\int_0^r x \,.\, x\,\sqrt{r^2 - x^2} \,.\, dx$$

Let, $x = u$; $x\sqrt{r^2 - x^2} = dv$; then the value of this integral becomes,

$$\int_o^r x^2 . \sqrt{r^2-x^2} . dx = -\frac{1}{3} . x . (r^2-x^2)^{\frac{3}{2}} + \frac{1}{3}\int_o^r (r^2-x^2)^{\frac{3}{2}} . dx,$$

$$= -\frac{1}{3} x (r^2-x^2)^{\frac{3}{2}} + \frac{1}{3}\int_o^r \sqrt{r^2-x^2} (r^2-x^2) dx;$$

from which by transposition,

$$\int_o^r x^2 \sqrt{r^2 - x^2} . dx = -\frac{x (r^2 - x^2)^{\frac{3}{2}}}{4} + \frac{r^2}{4} . \int_o^r \sqrt{r^2 - x^2} . dx$$

The first part of the right-hand member taken between the given limits, r and o, vanishes, and the second term,

$$\frac{r^2}{4} . \int_o^r \sqrt{r^2 - x^2} . dx = \frac{r^2}{4} \left[\frac{x \sqrt{r^2 - x^2}}{2} + \frac{r^2}{2} \sin.^{-1}\frac{x}{r}\right],$$

taken between the same limits, reduces to the single expression, $\frac{\pi r^4}{16}$.

Hence the resultant moment required will be

$$2 \frac{z_1}{x_1} \int_o^r x^2 \sqrt{r^2 - x^2} . dx = \frac{2 z_1}{x_1} . \frac{\pi r^4}{16}$$

$$= 2 z_1 . \frac{\pi r^3}{16};$$

which, being divided by the volume of the solid, equal to $\frac{2}{3} z^1 r^2$, as previously determined, gives the abscissa of the centre of gravity,

$$\bar{x} = \frac{3}{16} . \pi r .$$

VII. 7° *Example*:—Let it be required to find the centre of gravity of a conical or pyramidal solid, such as that shewn in Fig. 105. Assume the axis of, x, to pass through the vertex, O, and the centre of the base, B, and let the plane of the base be inclined to this axis at angle, θ. Resolve the inclined base and sections similar to it parallel to the vertical

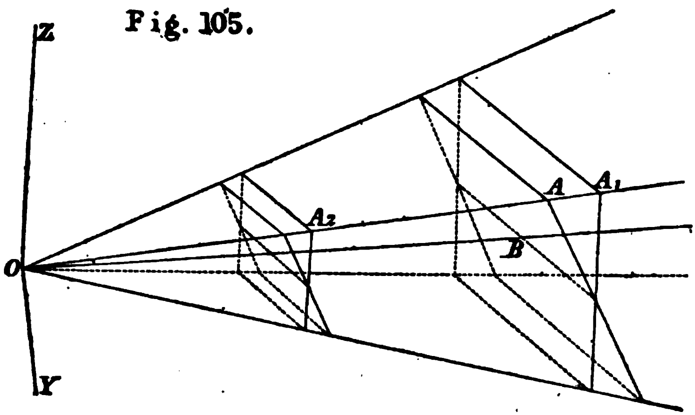
plane, YZ. Then if, S, represent the area of the inclined base and, S_1, that of its projection,

$$S_1 = S.\sin.\theta.$$

Let, x, y, z, be the co-ordinates of the point, A_2, of the projected figure, S_2. The area, S_2, will depend on the character of the form of base. If its form be rectangular,

$$S_2 = 4\,yz,$$

Fig. 105.



and the sum of the areas of all sections parallel to, S_2, will be

$$\int_0^h 4\,y\,z,$$

where, h, is the abscissa of the limiting or base-section, S_1. Now, h, y_1, z_1, being the co-ordinates of the point, A_1, of the end section,

$$\frac{y}{y_1} = \frac{x}{h} = \frac{z}{z_1},$$

from which relations it can be deduced that

$$y\,.\,z = \frac{y_1}{h}\,.\,x \times \frac{z_1}{h}\,.\,x = \frac{x^2}{h^2}\,y_1\,z_1;$$

wherefore,

$$4yz = \frac{x^2}{h^2} \cdot 4 y_1 z_1$$
$$= \frac{x^2}{h^2} \cdot S_1 = \frac{x^2}{h^2} \cdot S. \sin. \theta.$$

Consequently the volume of the whole solid will be,

$$\int_0^h 4yz\,dx = \int_0^h \frac{x^2}{h^2} \cdot S. \sin. \theta \,.\, dx = \frac{1}{3} \cdot h \,.\, S. \sin. \theta.$$

and the abscissa of the centre of gravity,

$$\bar{x} = \frac{\int_0^h x \cdot \frac{x^2}{h^2} \cdot S. \sin. \theta \,.\, dx}{\frac{1}{3} h.\ S. \sin. \theta} = \frac{3}{4} h.$$

In any other figure, having a different form of base, the area of any section, distant, x, from the vertex, bears to the area of the end section the constant ratio, expressed by, $\frac{x^2}{h^2}$. Hence, so long as the solid is of the same kind, a change of character in the base-form of the figure will not alter the nature of the investigation.

Any given rectilineal figure of uniform thickness can be subdivided into a series of triangles, the centres of gravity of which can be found by the usual rule, and thence the general centre of gravity of the whole figure, by supposing weights proportionate to the areas of the several triangles to act at their respective centres of gravity.

The centre of gravity of any frame-work, such as a roof-truss, can be found by first determining the centres of gravity of the component bars, and thence deducing the general centre by supposing weights proportionate to the dimensions of these bars to act at their respective centres of gravity.

Let the system of weights, P, Q, R, be situated at given

perpendicular distances, p, q, r; p', q', r'; p'', q'', r'', from the co-ordinate planes, yz, xz, xy; then the perpendicular distances, x, y, z, of the centre of gravity of the system of weights from these planes are given by the formulæ,

$$x = \frac{Pp + Qq + Rr}{P + Q + R}$$

$$y = \frac{Pp' + Qq' + Rr'}{P + Q + R}$$

$$z = \frac{Pp'' + Qq'' + Rr''}{P + Q + R}$$

EXAMPLES.

1. Find the height, $\bar{x}$, of the centre of gravity of the *Swansea Station Roof*, Fig 165, Pl. I., above the horizontal tie-rod, AA', assuming the weights of the various members, (1 – 24), to be proportional to the stresses induced in them.

$\bar{x} = 4{\cdot}6$ feet.

2. Taking the central longitudinal section of a locomotive as the plane of, xy, and the centre of the crank-shaft as origin, let us suppose that the weights of the separate parts have been reduced to a weight of 10 tons, concentrated at the point, defined by $x_1 = 6$ ft., $y_1 = 4$ ft., a second weight of 14 tons at a point, $x_2 = o$, $y_2 = 3$ ft., and a third, also equal to 14 tons, at a point, $x_3 = -7$ ft., $y_3 = 2$ ft.,—find the co-ordinates of the centre of gravity of the engine.

$\bar{x} = -1$ ft.; $\bar{y} = 2{\cdot}9$ ft.

3. Find the centre of gravity of a solid triangular cantilever, weighing 2 tons, projecting 36 ft.. and carrying at its outer end a suspended weight of 4 tons.

Centre, 8 ft. from outer end.

4. If a triangular entablature of uniform section be supported at each of its three corners by a vertical column, shew that the reactions at the columns are equal to each other.

5. Shew that the centre of gravity of any open triangular truss, of equal scantlings, coincides with the centre of the circle, inscribed in the triangle formed by joining the middle points of the side rafters and tie-rod of the given truss.

6. Given an angle-iron of the dimensions, $6'' \times 4'' \times \frac{3''}{8}$; shew that its centre of gravity lies on the circumference of a circle, described about its *bend*, or angular point, as a centre, of a radius equal to

$$r = 1{\cdot}97 \text{ in.}$$

7. A locomotive, weighing 30 tons, rests upon a turn-table, which for the moment we suppose supported at three points of its circumference, A, B, C, the lines joining which form an equilateral triangle and are each 20 ft. in length. If the weight of the engine be concentrated at a point, distant 2 ft. from each of the sides, AB and AC, what will be the reaction at the three points of support, A, B, C?

$$A = 22{\cdot}96 \text{ tons}; \; B = 3{\cdot}52 \text{ tons}; \; C = 3{\cdot}52 \text{ tons}.$$

8. Shew that, if, DE, be the arc of a circle, G, its centre of gravity, and, C, the centre of the circle,

$$CG = \frac{\textit{radius}, \, CD. \, \textit{chord}, \, DE}{\textit{arc}, \, DE}.$$

9. Shew that the centre of gravity of a circular ring, bounded by circles of the radii, R and r, respectively, and subtending an angle, 2θ, at the centre, C, of the circle, lies on a line bisecting the angle, 2θ, at a distance from the centre equal to

$$CG = \frac{2}{3} \cdot \frac{R^3 - r^3}{R^2 - r^2} \cdot \frac{\sin.\,\theta}{\theta}.$$

10. Shew that the centre of gravity of a segment of an arc, subtending an angle, 2θ, at the centre, C, of a circle of radius, r, lies on the line of bisection of the angle, at a distance from the centre equal to

$$CG = \frac{2}{3} \cdot \frac{r \sin.^3 \theta}{\theta - \sin.\theta \cos.\theta}.$$

11. Find the centre of gravity of the area contained between the curves, $y^2 = ax$, and, $y^2 = 2ax - x^2$, lying above the axis of x.

$$\bar{x} = a.\frac{15\pi - 44}{15\pi - 40}; \; \bar{y} = \frac{a}{3\pi - 8}.$$

12. In a section of T iron, the flange is of the dimensions, $b \times h$, and the web, $b_1 h_1$; shew that the distance, $\bar{x}$, of the centre of gravity of the section, from the top-line of the flange, is

$$\bar{x} = \frac{1}{2} \; \frac{bh^2 + b_1 h_1^2 + 2b_1 h h_1^2}{bh + b_1 h_1}.$$

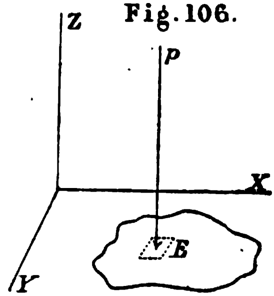
CHAPTER V.

MOMENTS.

1. BENDING MOMENTS :—Let a stress, p, Fig. 106, act normally against a surface, E, situated parallel to the plane of $X\,Y$. The stress distributed over an element of the surface will be equal to $p.\,dx.\,dy$, which, if $p = a.\,x$, may be put into the form, $a\,x\,d\,x.\,d\,y$. The moment of this elemental stress with respect to the axis of, x, can be expressed by

$$y.\,a\,x.\,dx\,d\,y,$$

Fig. 106.



and the sum of all such moments, comprised in the surface, by

$$M_x = a\iint x\,y\,dx\,dy = a.\,K.$$

In like manner the sum of the moments about the axis of, y, which are opposite in sign to those about the axis of, x, can be expressed by

$$M_y = -a\iint x^2.\,dx\,d\,y = -a.\,I.$$

Moreover, since the direction of stress is parallel to the axis of, z, there will be no moment about that axis. In that case the resultant moment is given by the formula

$$M = \sqrt{M_x^2 + M_y^2} = a \sqrt{K^2 + I^2}$$

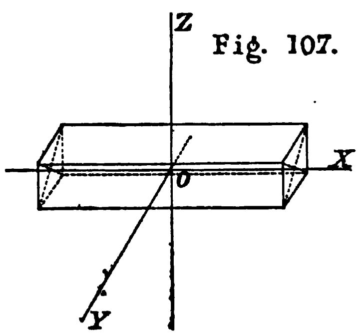
Let, λ, be the angle which the axis of the resultant couple makes with the axis of, x; and μ, that which it makes with the axis of, y; then,

$$cos. \lambda = sin. \mu = \frac{M_x}{M} = \frac{-K}{\sqrt{K^2+I^2}};$$

$$cos. \mu = sin. \lambda = \frac{M_y}{M} = \frac{-I}{\sqrt{K^2+I^2}};$$

$$tan. \mu = -\frac{K}{I}$$

Fig. 107.



If the figure submitted to stress be symmetrical with respect to the axis of, x, the summations in the sense of, y, on either side of the axis of, x, will balance each other, and cause the term, $K = \iint x\,y\,dx\,d\,y$, to vanish.

In that case, $M_x = O$, and $\mu = o$, and the only couple existing is that about the axis of y, viz., $M_y = -a\,I$. A beam, submitted to constant pressure per unit of length, and symmetrical in section and position with respect to the axis of, x, is in the condition of stress above stated. The only moment

created at any cross section of such a beam, made by a plane, $X\,Y$, Fig, 107, is that given by the expression

$$M_y = -\iint p.\, x\, dx\, dy.$$

This moment is taken, for the limits of the figure, on the right of the plane of section, and its effect would be to turn the beam round the axis of, y, in the direction pursued by the hands of a watch.

If, however, other forces act on the left of the plane of section, tending to turn the beam in an opposite sense, and equal in effect to those applied on the right of the sectional plane, the moment of this second system will be symbolically expressed by

$$-M_y = \iint p.x\, dx\, dy.$$

In this case the two moments, M_y and $-M_y$, balance each other, keeping the plane of section and the beam in equilibrium. When this happens, the moment induced at any cross-section can be determined by integrating the general form,

$$M_y = \iint px\, dx\, dy,$$

taken between the limits defined by the sectional plane and either extremity of the beam. As shewn above, the result is the same, whichever extremity be chosen as the limit of integration.

If, moreover, the forces are all applied in the central, longitudinal section of the beam, the moment of a force, p, applied at a point distant by, x, from the axis of, y, will be equal to, $p\, dx.x$; and if a series of forces, p, be distributed uniformly along the length of the beam, the sum of the elemental moments of those forces about the axis of, y, will be given by the expression,

$$M_y = \int p.\, x.\, dx.$$

Similarly, the sum of the distributed forces on the right of the section would be equal to,

$$F = \int p.\,dx.$$

Both of these integrals are taken between the limits defined by the cross-section and the right-hand extremity of the beam; though, as before stated, they might with equal correctness be referred to the left-hand limit of the beam.

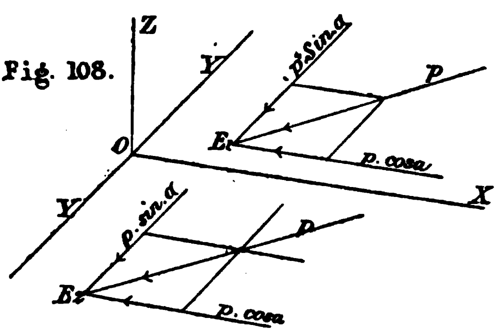
Let, x_o, be the abscissa of a point at which a resultant force may be supposed to act, producing the same moment at the cross-section as that due to the united action of the distributed forces; then by equality of moments,

$$F.\,x_o = M_r$$

$$x_o = \frac{M_r}{F} = \frac{\int p.\,x\,dx}{\int p\,dx}$$

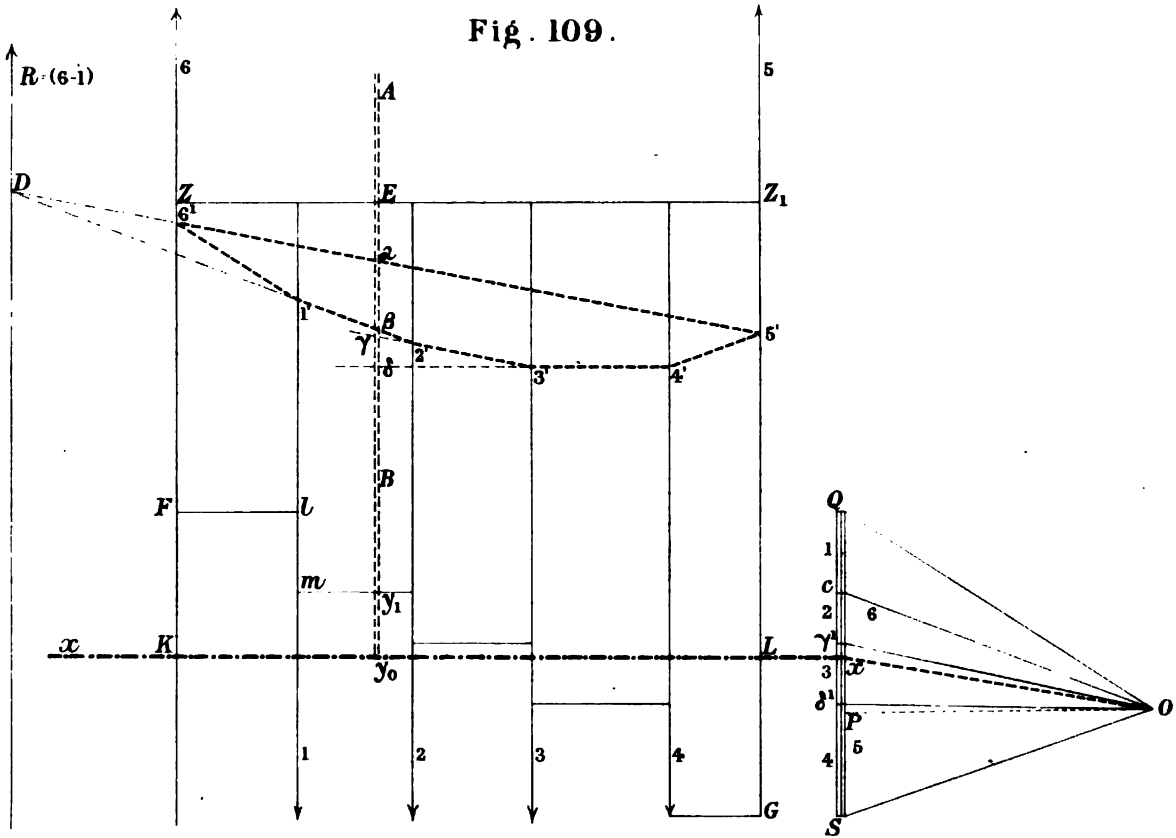
2. TWISTING MOMENTS:—If the force, p, Fig. 108, were to act tangentially instead of normally against the given plane

Fig. 108.



surface, it would have a tendency to turn the body about the axis of z, which in this case is supposed to be fixed. Let, E_1, Fig. 108, be the point of application of one of the forces, p, contained as stated in the plane of, $x\,y$; and let, a, be the angle which the direction of this force makes with the axis of, x.

Fig. 109.



...by Lockwood ...

Thos. Kell & Son, Lith

Decompose the force, p, into components, $p\ cos.\ a$ and $p\ sin.\ a$, parallel respectively to the axes of, x and y.

The component, $p\ sin.\ a$, will give rise to a moment about the axis of, z, expressed by,

$$M_1 = p\ sin.\ a\ dx.\ dy \times x$$

or, if $p = a\ x$,

$$M_1 = ax^2.\ dx.\ dy.\ sin.\ a = a.\ I.\ sin.\ a.$$

Similarly the component, $p\ cos.\ a$, will give rise to a moment about the same axis of revolution, given by the expression,

$$M_2 = -a\ x.\ y.\ dx.\ dy.\ cos.\ a = -a\ K.\ cos.\ a.$$

The moments, M_1 and M_2, tend *per se* to turn the body in different senses about the axis of, z; wherefore the total tendency to rotate the body about the central axis will be equal to the difference or algebraic sum of these two moments; that is

$$M_z = M_1 + M_2 = a\,[I\ sin.\ a - K\ cos.\ a].$$

The signs of the co-ordinates, composing the values of I and K, must be carefully attended to and distinguished; for, in certain cases and for special points of application, such as E_2, Fig. 108, the component moments tend *per se* to turn the body in the same sense about the axis of, z.

3. Graphic Delineation of Shearing Forces and Bending Moments:—Shearing Force is a force acting tangentially at a section of a beam and exercising an action similar to that of a pair of shears. It has been previously shewn that the shearing stress at any section of a beam can be found by taking the sum of the forces applied between the given section and the right or left extremity of the beam [§. 1]. The result was proved to be the same, whichever end of the beam were chosen as the limit of integration.

Let, therefore, a section be made by a dividing plane, $\overline{A\,B}$, Fig. 109, through a given beam, $\overline{Z\,Z_1}$. The shearing force acting

at the plane of section, $\overline{A\,B}$, will be equal to the sum of the forces acting between, $\overline{A\,B}$, and the left extremity, Z, of the beam. This sum, is, therefore, given by the algebraic sum of the forces, 6 and 1, which, as they act in opposite directions, will be equal to their difference,

$$[6-1] = x\,C, \text{ on the polygon of forces.}$$

It has been shewn earlier in this work, [Pt. I. Ch. I. § 3], that the point of intersection of the first and last lines drawn in the polar polygon of a system of forces determines a point in the line of action of the resultant of that system. Consequently, in order to find an objective point in the resultant path of the forces, 6 and 1, it is necessary to construct a special polar polygon for those forces. It will be seen that this special polygon forms part of the general polar polygon of the system, marked, $1'\,2'\,3'\,4'\,5'\,6'$, in Fig. 109. For, the part, $x\,Q\,c$, on the line of forces, is the graphic representation of the forces, 6 and 1. Hence, the first line drawn in the special polygon will correspond to the line, $5'\,6'$, parallel to the polar line, $O\,x$; the second line will in like manner correspond to the line, $6'\,1'$, parallel to, O_{61}; and the last line drawn will coincide with, $1'\,2'$, made parallel to, O_{12}. The first and last lines, $5'\,6'$ and $1'\,2'$, will, if produced, intersect in a point, D, to the left of the greater and negative force, 6. The resultant, $R = [6-1]$, of the two forces will pass through the point, D, and be directed upwards in the same sense as the greater force, 6.

Let two forces equal in magnitude to, R, and of opposite signs, be added to the system and supposed to act in the line of section, $\overline{A\,B}$. These two forces will mutually balance and destroy each other, and will not therefore disturb the existing state of equilibrium. One of these forces, equal in magnitude and direction to the resultant, R, will constitute what is called the *shearing force* at section, $\overline{A\,B}$.

The remaining forces, R, upward at D, and R, downward at, E, will form a couple,

$$R \times \overline{D\,E}$$

This couple is called the *bending moment* at the given section.

Now the triangles, $D\,\alpha\,\beta$ and $O\,c\,x$, are similar;—consequently their bases and altitudes are to each other in the same ratio; that is

$$\frac{x\,c}{O\,P} = \frac{\alpha\,\beta}{D\,E};$$

but, $\overline{x\,c}$ = [6-1] = R;—therefore

$$R \times \overline{D\,E} = \alpha\,\beta \times \overline{O\,P}.$$

Hence, if, $\overline{O\,P}$, be taken as the unit of scale;

$$R \times \overline{D\,E} = \alpha\,\beta,$$

from which it may be inferred that the intercept, $\alpha\,\beta$, on the trace of the plane of section, $\overline{A\,B}$, cut off by the closed polar polygon of the system of forces, graphically represents the bending moment at the given section.

With regard to the graphic representation of the shearing force, it is only necessary to construct a figure, such that the line, $\overline{x\,x}$, being taken as the initial line, ordinates, $\overline{y_0\,y_1}$, may be intercepted on the traces of the section-planes, representative of the shearing forces at the different parts of the beam considered. Since the shearing forces are constant for certain lengths of the beam, and always equal to the algebraic sum of special forces, shewn graphically on the polygon of forces, it is evident that the figure of shearing forces must take a stepped form. The various steps of the figure will be determined by the intersections of horizontal lines, drawn through the extremities of forces, represented on the line or polygon of forces, with their corresponding lines of action. Thus, the horizontal lines, $\overline{Q\,F}$ and $x\,K$, bounding the force, 6, on the line of forces, meet the line of action of the same force in points, F and K, and determine one step of the required figure. In like manner, lines, $Q\,l$ and $c\,m$, meet the line of action of force, 1, and establish a second step of the

figure. Similarly, the other steps can be found, and the polygon of shearing-forces, $F K L G$, constructed.

For the plane of section, $\overline{A B}$, the shearing force is given by the intercept, $y_0 y_1$, which, it will be seen, is equal to the part, $x C$, on the line of forces.

4. MOMENTS OF FORCES WITH RESPECT TO POINTS:—Let it be required to find the moment of a force, F, Fig. 110, about a point, C, or in more precise terms, about an axis traversing, C, at right angles to the plane of the paper.

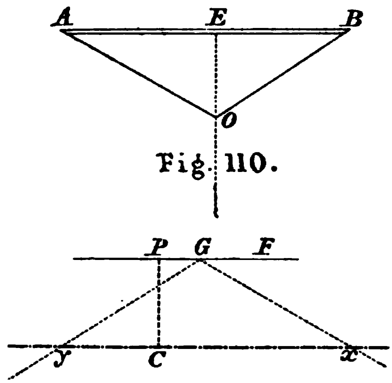


Fig. 110.

Draw a line, $\overline{A B}$, parallel to the direction of the given force, representing it in magnitude and direction.

From any point, E, in the line, $\overline{A B}$, erect a perpendicular, $\overline{E O}$, and assume its length to be the arbitrary unit of scale. Join, O, with the extremities, A and B.

Take any point, G, in the objective line of action of the force, F, and from, G, draw two lines, $G x$ and $G y$, parallel respectively to, $\overline{O B}$ and $\overline{O A}$. Through C, draw a line parallel to the direction of, F, determining an intercept, xy, between the lines, Gx and Gy. This part, xy, cut off, will measure the moment of the force, F, about the point, C, according to the scale constructed upon the base, $\overline{E O}$, as unit of length. For since the triangles, $O A B$, and Gxy, are similar, their bases are to each other as their altitudes; or

$$\frac{A B}{E O} = \frac{x y}{C P};$$

therefore,

$$\overline{AB}\,.\,\overline{CP} = \overline{xy}\,.\,\overline{EO}$$

But, $\overline{EO}$, has been made the unit of scale ; hence,

$$\overline{AB}\,.\,\overline{CP} = \text{Force} \times \text{arm} = \overline{xy}.$$

5. MOMENTS OF THE RESULTANTS OF FORCES WITH RESPECT TO POINTS:—The law just explained for the moment of a single force about a given point holds equally in the case where it is required to find the moment of a given system of forces about a given point. Let the system be that shewn in Fig. 112, its corresponding polygon of forces being represented in Fig. 111. The closing line, $\overline{AE}$, of the latter

Fig. 111.

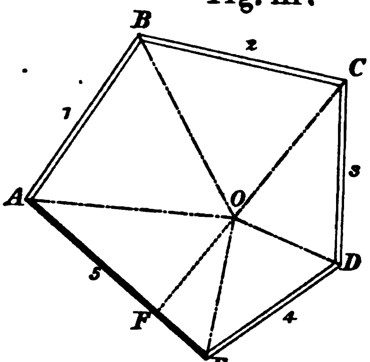


Fig. 112.

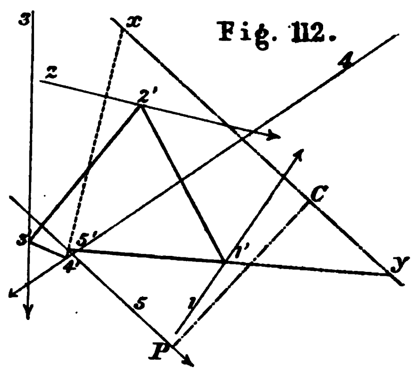


figure will define, both in magnitude and direction, the resultant of the forces, 1, 2, 3, and 4. [Pt. I. Ch. I. § 1.]

Construct the polar polygon, $1'\,2'\,3'\,4'\,5'$, of the given system relatively to the pole, O. The point, $5'$, determined by the intersection of the lines, $1'\,5'$ and $4'\,5'$, will fix a point on the objective path of the resultant force, and a line drawn through, $5'$, parallel to, $\overline{AE}$, will describe the path itself.

Through, C, the given point about which moments have to be taken, draw a second line parallel to, $\overline{AE}$. Produce the extreme lines, $5'\,1'$ and $5'\,4'$, of the polar polygon, determining upon the line through, C, an intercept, $\overline{xy}$. The part, $\overline{xy}$,

cut off, will measure the moment of the resultant of the system about the given point. For, since the triangles, $O\,A\,E$ and $5'\,x\,y$, are similar, their bases are to each other as their altitudes; hence,

$$\frac{A\,E}{O\,F} = \frac{x\,y}{C\,P}\text{; or, } \overline{A\,E}\,.\,\overline{C\,P} = \overline{x\,y}\,.\,\overline{O\,F}$$

Now, let, $\overline{O\,F}$, be taken as the unit-length of scale; then

$$\overline{A\,E}\,.\,\overline{C\,P} = \text{Resultant-Force} \times \text{arm} = \overline{x\,y}.$$

6. Moments of Applied Loads about points in Arched Ribs:—Let it be required to find the bending moment, relatively to a point, $a°$, contained in a normal section of an arched rib, having the form given in Fig. 113 and subject to a definite series of loads and reactions, marked (1), (2), (3), (4), (5), (6), (7), and (8).

Construct the polygon of forces as given in Fig. 114, the lines, 7 and 8, representing the inclined reactions at the abutments.

According to a principle formerly enunciated [Pt. III. Ch. I. § 1], the resultant force acting at the normal section, $\overline{A\,B}$, will arise from the composition of the forces applied between that section and either extremity of the arched rib. Choose those lying on the left, viz., (1), (2) and (8). Now, the part, $X\,A\,c$, of the double-lined polygon of forces, Fig. 114, is the graphic representation of the forces, (8), (1), and (2);—wherefore the line, $X\,c$, joining the loose ends, X and c, will represent in direction and magnitude the resultant force acting at the section, $\overline{A\,B}$. The problem is now reduced to that of finding the moment of the resultant, $X\,c$, about the given point, $a°$, which can be done according to the general rule given in § 5.

Namely;—construct first the special polar polygon, $8_0\,1_0\,2_0\,c_0$, determining thereby a point, c_0, (lying within the limits of the figure), on the objective path of the resultant of the forces, (8), (1), and (2). Secondly, from any point, E, in $X\,c$, erect a perpendicular, $\overline{E\,O_1}$, equal in length to the unit of scale, and join, $X\,O_1$, $c\,O_1$. This unit of scale, which is an

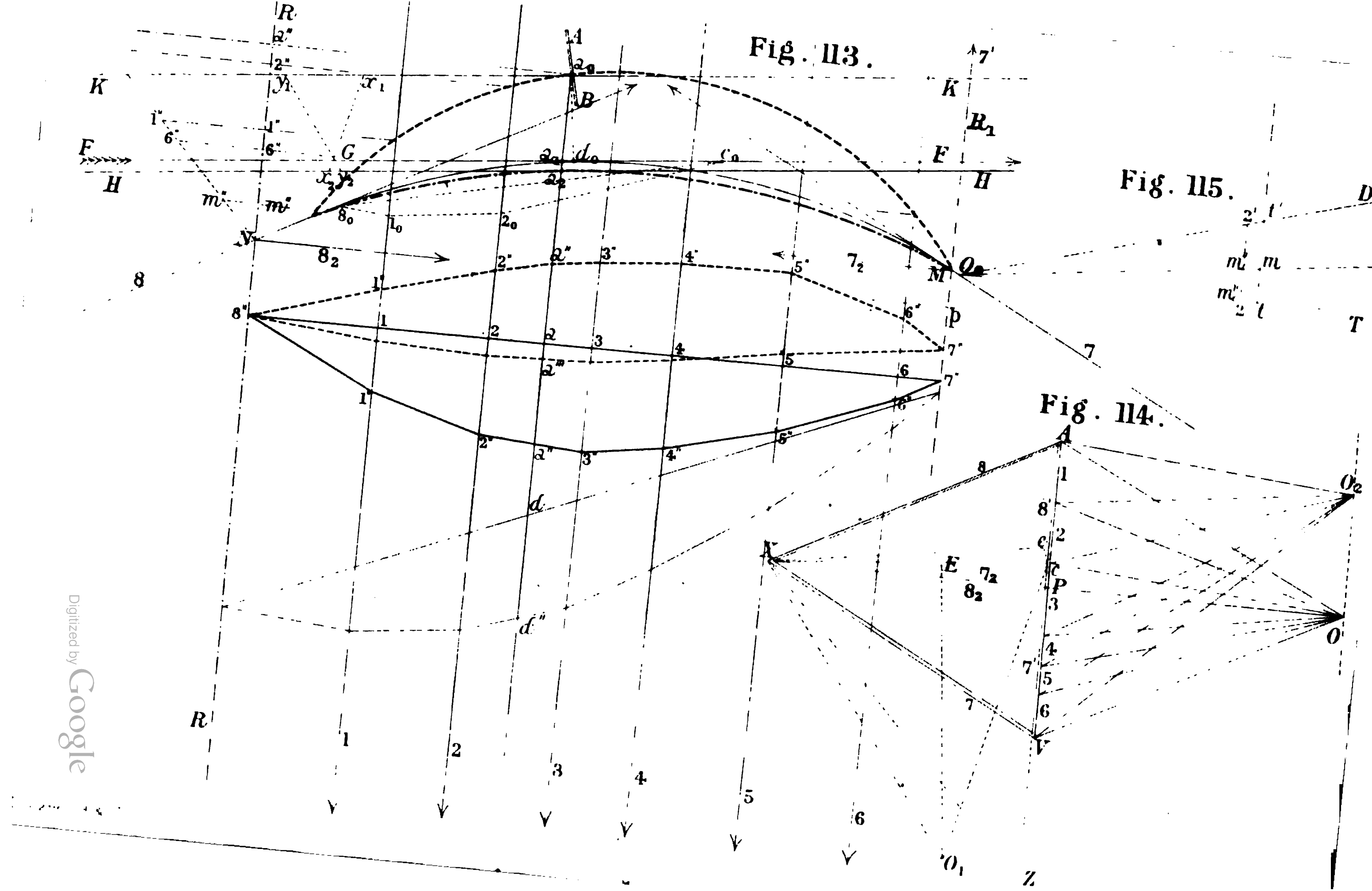


Fig. 113.

Fig. 114.

Fig. 115.

arbitrary unit, has in this case been made equal to the perpendicular, $\overline{OP}$, let fall from the pole, O, on the line of forces.

Thirdly, through the points, a°, and c_0, draw lines parallel to $X c$; and from any point, G, on the resultant line of action through, c_0, set off two lines, $G x_1$ and $G y_1$, parallel respectively to, $c O_1$ and $X O_1$. The intercept, $x_1 y_1$, cut off by these two lines on the line through, a°, will determine the moment required.

It is important to notice that the position of, $\overline{FF}$, the resultant force of the system, (8), (1), (2), is independent of the form of arch or other structure to which those three forces may be applied. The moment about the point, a°, will remain the same, no matter how the form of the structure may be varied to which it may belong, provided that it represent the same fixed point in space.

Draw a vertical line through the point, a°, and let it intersect the line of action of the resultant, $\overline{FF}$, in a point, a_1. Let a_2, be a third point on the vertical line, $a_1 a^\circ$. Construct, as shewn, two forms of arch passing respectively through the points, a_1 and a_2

For sections made through the three forms of arch passing through points, a°, a_1, a_2, placed on the same vertical line, it will be seen that the bending moments vary in nature and amount. The bending moment relatively to the point, a°, situated in the first type of arch, is graphically represented by, $\overline{x_1 y_1}$, and tends to turn the part-rib of the arch from right to left. The moment relatively to the point, a_2, situated on the third form of arch, is graphically equivalent to $\overline{x_2 y_2}$, and has a tendency to turn the part-rib of the arch from left to right; or in other words to crush it in. Lastly, the moment about, a_1, —a point common to the neutral fibre of the second type of arched rib and the line of action $\overline{FF}$, of the resultant force,—vanishes.

These moments can be found by three different graphic methods. First, in the way just explained by the construction of the special polar polygon, $8_0\, 1_0\, 2_0\, c_0$. Secondly, by the graphic construction of the product of the resultant force acting along, $\overline{FF}$, and the arm or perpendicular distance,

$a_0 d_0$, of any point, a_0. The constructive process necessary for this purpose is given in Fig. 115. Set off along the horizontal line, $\overline{O_0 T}$, a distance, $O_0 t$, equal to the unit-length of scale, $\overline{OP}$. Along a line, $t t'$, at right angles to, $\overline{O_0 T}$, lay off a distance, $t t'$, equal in length to the *arm* of the resultant force; or to $a_0 d_0$.

Make the length, $O_0 2$, equal to, $X c$, the magnitude of the resultant of the system of forces, (8), (1), (2); and from the point, 2, so determined erect a perpendicular, 22′, meeting the line, $O_0 t'$, in a point, 2′; then by similar triangles,

$$\frac{22'}{t t'} = \frac{O_0 2}{O_0 t},$$

that is,

$$22' \times O_0 t = O_0 2 \times tt'.$$

But, $O_0 t = \overline{OP} =$ unit-length of scale, therefore,

$$\begin{aligned} 22' &= O_0 2 \times tt' \\ &= X c \times a_0 d_0 \\ &= \text{magnitude of resultant} \times \text{arm} \\ &= \text{moment about } a_0 \end{aligned}$$

Since the lines, 22′ and, $\overline{x_1 y_1}$, both represent one quantity, they must be equal in length, which fact can be used as a check upon the two methods employed.

But the value of the same moment can be found in a third way. Decompose the system,

(1), (2), (3), (4), (5), (6), (7), (8)

into two series of forces, horizontal and vertical. The vertical series is represented on the polygon of forces, by the divisions (1), (2), (3), (4), (5), (6), and the two equal reactions at the abutments, (7′) and (8′). The horizontal components are two equal and opposite thrusts applied at the abutments and represented in magnitude by the line, $\overline{XP}$, on the polygon of forces. Name these thrusts respectively, (7_2) and (8_2).

The moment about, a_0, due to the forces, (8), (1), (2), on the

left of the plane of section, can be then found by determining the separate moments arising from their vertical and horizontal components, and taking the graphic sum.

To find the moment about a_0, due to the vertical series construct the polar polygon of the vertical forces,

(1), (2), (3), (4), (5), (6), (7'), (8').

This can readily be done according to the principles and method explained in, Pt. I. Ch. I. § 2.

Namely, draw any vertical line, $\overline{RNR}$, meeting the objective path of the reaction, (8), in a point, N. From any point, 8″, on the line, $\overline{RNR}$, set off two lines (in full), 8″ 1′ and 8″ 7″; parallel respectively to the polar lines, $O_{18'}$ and $O_{7'8}$, on the polygon of forces. From point, 1″, where the line, 8″ 1″, intersects the force line, (1), draw a line, 1″ 2″, parallel to the polar line, O_{12}. From, 3″, draw a line, 2″ 3″, parallel to, O_{23}; and so on as far as the point, 6″. From, 6″, draw a line, 6″ 7″, parallel to the polar line, $O_{67'}$, meeting the line, 8″ 7″, first drawn in a point, 7″. The point, 7″, so determined will furnish a point on the objective path of the vertical reaction, 7′. Through 7″, draw the vertical line, 7″ R_1, intersecting the path of the abutment pressure in a point, M. This point, M, will represent the proper *locus* of the reactions at the right-hand abutments, correlatively to the point of application, N, chosen for the left-hand abutment pressures, on the assumption made, that the horizontal and vertical reactions at one end of the arched rib are equal to the corresponding reactions at the other;—a condition implicitly assumed in the construction of the polygon of forces.

Through the point, a_0, draw a vertical line determining an intercept, aa'', in the polar polygon just constructed. According to what has already been demonstrated [§ 3.], this intercept is the graphic representation of the moment induced at, a°, by the given system of vertical forces.

It remains to find the partial moment about, a°, due to the thrust, 8_2, acting in a horizontal direction through the point, N. This can be done by the rule given in § 4, for finding

the moment of a known force about any fixed point in space. Namely, set off from the line, $\overline{XP}$, representing the intensity of horizontal thrust, a perpendicular distance, $\overline{PZ}$, equal to the unit-length of scale, $\overline{OP}$. Join $\overline{XZ}$, $\overline{ZP}$, and from, N, draw two lines, $\overline{NQ}$ and $\overline{NR}$, respectively parallel to, $\overline{XZ}$ and $\overline{ZP}$.

From a°, draw a horizontal line parallel to the direction of the thrust, determining an intercept, $a''a''$, between the lines, $\overline{NQ}$ and $\overline{NR}$. This intercept gives the graphic value of the moment of the force, 8_2, about the point, a°.

Now, comparing the directions of the forces, (8') and (8_2). with respect to the position of the point, a°, it is evident that the separate moment due to the vertical system of forces has a tendency to turn the part-rib, from left to right, or to crush it in; whilst that due to thrust has an opposite tendency; namely to turn it from right to left, or to lift up the end, a°.

Consequently, the graphic sum of the separate moments will be equal to the difference in length of the lines, $a''\,a''$, contained between, $\overline{NQ}$ and $\overline{NR}$, and the intercept-line, aa'', in the polar polygon of vertical forces. Hence, if from the point, a'', on the polar polygon a distance be set off vertically, equal to the intercept, $a''\,a''$, cut off by lines, $\overline{NQ}$ and $\overline{NR}$, the resultant moment will be given by the graphic sum,

$$a''\,a = a''\,a'' - a\,a''.$$

In a similar way other intercepts, such as, 1″ 1″, 2″ 2″, 3″ 3″, &c., have been determined between the lines, $\overline{NQ}$ and $\overline{NR}$, by drawing horizontal lines through the points of intersection of the neutral fibre of the uppermost arched rib with the lines of action of the applied forces. These distances have been set off vertically, starting from the corresponding points, 1″, 2″, 3″, &c., on the polar polygon, so as to determine the graphic sums or moments, 1″ 1, 2″ 2, 3″ 3, &c., relatively to the before-mentioned points of intersection of the arch and force lines.

The upper row of points, 1″, 2″, 3″, ... 8″, being connected together by a series of lines, 8″ 1″, 1″ 2″, 2″ 3″, ...

6″ 7″, the closed polygon, 8″ 7″ 7″ 8″, thus completed will enable the moment to be found relatively to any point whatsoever of the uppermost arch, by simply drawing a vertical line through the selected point, and then taking the graphic sum intercepted between the horizontal line, 8″ 7″, and the *upper limits* of the closed polygon, 8 7″ 7″ 8. For example, at the point where the force-path, (6), crosses the neutral fibre of the uppermost arch, the moment induced by the given system of forces will be graphically equivalent to, 6″ 6.

Similarly, the length, 5″ 5, will be a graphic measure of the moment about the point, where the force-path, (5), intersects, the neutral fibre of the arched rib; and at the right hand abutment there will exist a moment about the point, *M*, represented by, 7″ 7″, and so on for other points.

It will, therefore, be seen that the polar polygon, 8″ 7″ 7″ 8″, contains the graphic expressions of three kinds of moments. The full intercepts, such as, 1″ 1″, 2″ 2″, *a″ a″*, &c., give the graphic values of moments due to the horizontal forces, or thrusts; the partial intercepts, 1″ 1, 2″ 2, *a″ a*, &c., situated *below* the line, 8″ 7″, determine the moments due to the vertical system of forces; and the intercepts, 1″ 1, 2″ 2, *a″ a*, &c., situated *above* the same line, are measures of the graphic sums or moment-resultants arising from the two systems of forces taken together.

It may be useful to draw attention to an important result of the preceding construction. The points, *M* and *N*, representing the *loci* of abutment pressures, do not lie on the same level. This arises from the fact that the vertical load is unequally distributed over the arched rib, as well as from the implied condition that the horizontal thrusts and vertical reactions at the abutments must be separately equal. As a consequence of this difference in level of the points, *M* and *N*, a moment, due entirely to thrust, will be induced at the right end abutment. The graphic value of this moment can be found by drawing through, *M*, a horizontal line so as to determine the intercept, *m″ m″*, cut off as usual by the lines, $\overline{NQ}$ and $\overline{NR}$. This moment is, as just stated, entirely due to the fact that the equal horizontal reactions at, *N* and *M*, act at

different levels and are not therefore directly opposed. The moment in question is in reality equal to the common intensity of horizontal thrust multiplied by the arm or perpendicular distance, Mp. The graphic construction of this product is given in Fig. 115; where,

$$O_o t = \text{unit-length of scale} = \overline{OP}$$
$$O_o m'' = \text{horizontal thrust} = \overline{XP}.$$
$$t\,m = \text{arm} = Mp.$$

Hence, by similar triangles,

$$\frac{m''m''}{m\,t} = \frac{O_o m''}{O_o t}$$

or,

$$m''m'' \times O_o t = O_o m'' \times m\,t \,[\text{or, since } O_o t = 1],$$
$$m''m'' = \text{thrust} \times \text{arm} = \text{moment required}.$$

The direction of this moment is left-handed, that is, it tends to turn the arched rib from right to left, and to lift the end, M, upon the hinge, N, as a fulcrum. This tendency can be counterbalanced by the addition of a weight at the right abutment of such an amount and leverage as to produce a positive moment equal absolutely to, $m''m''$, but having the effect of pressing down the end, M, of the arched rib.

The polar polygon of moments corresponding to the lowest type of arch is given in thinner dotted lines, the part-polygon connected with the vertical system of forces remaining the same. It will be seen that the graphic sums, or resultant moments, have less absolute values for this than for the higher form of arch. For example the moment-resultant, $a''a$, corresponding to the upper type of rib, is much greater than the moment, $a'''a$, induced at the fixed point, a_2, on the neutral fibre of the lower form. The part, $a\,a''$, below the horizontal line, $8''\,7''$, and due to the vertical system of forces, is the same for both ribs. It may be added that moments, $a''a$, above the line $8''\,7''$, are of opposite sign to moments, $a\,a'''$, measured below the same datum-line.

It is often useful to know the normal stress and shearing force brought to bear at any point in an arched rib, subject to a given series of forces,

(1), (2), (3), (4), (5), (6), (7), (8).

Let the point, a°, Fig. 113, be chosen. It has been shewn in the course of the preceding investigation that the resultant force of the given system, relatively to the section, $\overline{A\,B}$. is graphically represented on the polygon of forces by the line, $X\,c$, which determines it both in magnitude and direction. The objective path, in which this resultant acts, has been found to lie along, $\overline{F\,F}$.

Now the state of equilibrium in which the rib exists will not be altered by supposing two equal and opposite forces of the magnitude, $X\,c$, to act at, a°;—for, being opposed, they must mutually balance each other. One of these added forces, in combination with the actual resultant acting along, $\overline{F\,F}$, gives rise to the resultant moment about, a°, already fully considered. The remaining force at, a°, equal in every respect to the resultant force, $\overline{X\,c}$, with the exception that it acts at, a°, instead of in the path, $\overline{F\,F}$, can be resolved into two component forces, one parallel, the other perpendicular to the normal section, $\overline{A\,B}$. The former is what has been called the shearing force, the latter the normal stress at the same section.

If, in the polygon of forces, upon, $X\,c$, as a base, a triangle be constructed with its sides, $X\,e$ and $c\,e$, respectively perpendicular and parallel to the normal line of section, $\overline{A\,B}$; $\overline{e\,c}$, will give the shearing force, and, $X\,e$, the normal stress required.

The treatment of arched ribs, developed in this article, may be concluded by two observations;—1° the representative value of the unit of scale, $\overline{O\,P}$, must be expressed in lengths and forces measured by the line, $\overline{A\,V}$, on the polygon of forces, which is equal in all cases to the known sum of the vertical loads applied;—2°, the point, 7″, situated on the horizontal line, 8″ 7″, has been determined by the construction

of the polar polygon of the vertical series of forces, and the discovery of the point, 7″, leads at once to the localisation of the left-hand abutment pressures at M, which lies at the point of intersection of the force-path, (7), with a vertical line drawn through, 7″. The point, 7″, should therefore be carefully fixed by the construction of several polar polygons related to different poles. Two such polygons are given in the figure, one referred to the pole, O, the second to the pole, O_2. Both concur in determining points on the same vertical line through M, and both agree in giving the same value, $a\,a'' = d\,d''$, for the bending moment about, a°, due to vertical forces only.

It should also be remarked that the reactions have been supposed to take place in paths, (7) and (8), equally inclined to the horizon. Were this not the case, the construction, whilst remaining exactly the same in description, would introduce a few modifications in the results obtained;—insomuch that the horizontal line, $\overline{X\,P}$, would cease to bisect the vertical line of loads, $\overline{A\ V}$, and as a consequence the vertical reactions at the two ends of the rib would cease to be equal. But the horizontal thrust would still be the same at one end of the arch as at the other. The line, 8″ 7″, would no longer be horizontal, unless the position of the pole, O, were changed accordingly; so as still to lie upon the line, $\overline{X\,P}$, produced.

7. MOMENTS OF INERTIA.—Let it be required to find the moment, relatively to the plane-section, $\overline{A\,B}$,* of certain of the loads applied to the beam, $\overline{Z\,Z_1}$, Fig. 109, taken independently of the others; or, in the limit, the moment induced by one force only, such as, (3).

The latter moment can be determined by producing the two sides, 4′3′ and 2′ 3′, issuing from the point, 3′, on the polar polygon, until they intercept a part, $\gamma\,\delta$, on the produced trace of the sectional plane, representing the moment required. For, since the line, 2′ 3′ is drawn parallel to O_{23}, or $O\,\gamma'$, and the line, 3′ 4′, parallel to O_{34} or $O\,\delta'$, it can be easily shewn that the two triangles, $3'\,\gamma\,\delta$ and $O\,\delta'\,\gamma'$, are similar, and that consequently their bases bear the same ratio to each other as

* That is, relatively to an axis traversing any point in the line, $\overline{A\,B}$, perpendicularly to the plane of the paper.

their altitudes ;—that is, if, x_3, represent the perpendicular distance of the point of application of force, (3), from the trace, $\overline{A\,B}$,

$$\frac{\delta\,\gamma}{x_3} = \frac{\delta'\,\gamma'}{\overline{O\,P}}$$

or, since, $\overline{O\,P}$, has been made the scalar unit,

$$\delta\,\gamma = \delta'\,\gamma' \times x_3$$
$$= \text{Force (3)} \times \text{arm} = \text{Moment required.*}$$

In the same way, if it were required to find the moment relatively to, $\overline{A\,B}$, of the loads, (1), (2), (3), (4), looked upon as independent forces ; it would only be necessary to produce the extreme lines, $6'\,1'$ and $5'\,4'$, of the polar polygon, which would intercept on the trace, $\overline{A\,B}$, a part representing the moment required.

This being understood, let the forces, given in Fig. 109, and marked, (1), (2), (3), (4), (5), (6), be termed

$$F_1, F_2, F_3 \,.\,.\,.\,.\,.\,. \,F_6,$$

the magnitudes of the forces being given as before on the polygon of forces annexed. Moreover, let the perpendicular distances of the points of application of these forces from the trace, $\overline{A\,B}$, be represented by

$$x_1, x_2, x_3 \,.\,.\,.\,.\,.\,. \,x_6,$$

Then, according to a former demonstration [§ 3] ; or simply by inspection, it will be seen that

$$\alpha\,\beta = F_1\,x_1 + F_2\,x_2 + F_3\,x_3 + \,.\,.\,.\,.\,. \,F_6\,x_6,$$

which relation contains the graphic expression of a theorem, sometimes called the "*Theorem of the Superposition of Forces,*"

* The method of treating Moments of Inertia here expounded is based on a similar treatment of this part of the subject by Mons. Lévy, (Statique Graphique).

meaning that, as shewn in the above equation, the moment at any section, due to a given system of forces, taken as a whole and represented by one quantity, $\alpha\beta$, is equal to the graphic sum of the moments of the forces, separately taken, and referred to the same plane of section.

Again, let,

$$F^1 = F_1 x_1 \,;\; F^2 = F_2 x_2 \,;\; \ldots\ldots F^6 = F_6 x_6,$$

and suppose further the graphic values of these separate moments, F^1, F^2, . . . F^6, to have been determined by the same method as that employed above in finding the moment,

$$F^3 = F_3 x_3 = \delta' \gamma' \times x_3 = \delta \gamma.$$

Moreover, let a *plus* or *minus* sign be attached to each of these partial moments, accordingly as it tends to turn the plane of section in one or the other direction. Let the moments, thus determined both in magnitude and direction, be looked upon as so many new forces, applied at the same points as were the original loads, F_1, F_2 . . . . F_6. Treat these new force-moments in the same way as the original forces; that is, construct their corresponding polar polygon and polygon of forces. The limits of the new polar polygon will determine upon the trace of the sectional plane, $\overline{A\,B}$, an intercept, $\alpha_m\beta_m$, such that

$$\alpha_m\beta_m = F^1 x_1 + F^2 x_2 + \ldots\ldots F^6 x_6,$$

or since,

$$F^1 = F_1 x_1 \,;\; F^2 = F_2 x_2 \,;\; \ldots\ldots F^6 = F_6 x_6$$
$$\alpha_m\beta_m = F_1 x_1^2 + F_2 x_2^2 + \ldots\ldots F_6 x_6^2. \qquad \text{(See Note.)*}$$

The type-form of the separate terms of this sum can be represented by, $F_i x_i^2$, hence

$$\alpha_m\beta_m = \Sigma F_i x_i^2,$$

* Here the co-ordinate arms, or abscissæ, x_1, x_2, x_3 . . . x_6, are taken absolutely without regard to sign. The moment of inertia of a force takes the sign of that force.

which shews that the intercept, $\alpha_m \beta_m$, will graphically determine the sum of the products of the given forces, F_i, by the square of the perpendicular distances, x_i, between the section plane and the points of application of the different forces. The graphic sum, $\alpha_m \beta_m$, is technically termed the *Moment of Inertia of the given system of forces relatively to the axis*, $\overline{A\,B}$.

The moment of inertia, expressed by any of the separate terms, such as, $F_3\, x_3^2$, can be found in a similar way to that given for finding the moment of one of the single forces of a system, by simply producing the lines of the polar polygon, which meet on its objective path of application, till they cut off an intercept on the trace of the sectional plane, $\overline{A\,B}$.

If the given forces, having the same points of application, were horizontal instead of vertical in direction, it might be necessary to find their resultant moment of inertia relatively to an axis, $\overline{Z\,Z_1}$, situated anywhere in the plane of forces. This can be done in exactly the same way as for vertical forces, ordinates, y, replacing abscissæ x. The general forms of the expressions would not vary.

It will be different, however, if after having found the graphic sum of a series of separate moments, such as, $F^3 = F_3\, x_3$, it were required to find the moment of inertia, or the moment of these moments, considered as so many new forces applied at the same points as the original loads, and referred to the trace of any horizontal section plane, $\overline{Z\,Z_1}$, as an axis. In this case the terms composing the new graphic sum would be of the form,

$$F^3 \times y_3 = F_3 \cdot x_3 \cdot y_3$$

in which expression, y_3, represents the ordinate of the point of application of the force, F_3; or, its perpendicular distance from line, $\overline{Z\,Z_1}$. It would then be necessary to consider the series of force-moments,

$$F^1, F^2, F^3 \; . \; . \; . \; . \; F^n,$$

to be applied in a horizontal direction through the same points of application as the original forces, and to construct the

corresponding polar polygon. The trace of the section-plane, $\overline{ZZ_1}$, would cut off an intercept in this polar polygon, which we may name, $\alpha_n \beta_n$, giving the graphic sum, or moment of moments required, about any axis perpendicular to the plane of the paper and traversing any point situate in the line, $\overline{ZZ_1}$. such as, E, which is common to lines, $\overline{AB}$ and $\overline{ZZ_1}$. Here again the Theorem of the Superposition of Forces applies, so that,

$$\begin{aligned}\alpha_n \beta_n &= F^1 y_1 + F^2 y_2 + \ldots\ldots + F^6 y_6 \\ &= F_1 x_1 y_1 + F_2 x_2 y_2 + \ldots + F_6 x_6 y_6 \\ &= \Sigma F_i x_i y_i\end{aligned}$$

Following the same method and reasoning, and making

$$F_{\prime}^1 = F_1 x_1^2 ;\ F_{\prime}^2 = F_2 x_2^2 ;\ \ldots\ F_{\prime}^6 = F_6 x_6^2,$$

and

$$F_{\prime\prime}^1 = F_1 x_1 y_1 ;\ F_{\prime\prime}^2 = F_2 x_2 y_2 ;\ \ldots\ F_{\prime\prime}^6 = F_6 x_6 y_6,$$

the moments of the third order can be constructed, the general forms of which are,

$$\Sigma.\ F_i x_i^3,\ \Sigma.\ F_i x_i^2 y_i,\ \Sigma.\ F_i x_i y_i^2$$

In a similar way moments of a still higher order might be found; but in practical questions, those only of the second order; namely

$$\Sigma F_i x_i^2 \text{ and } \Sigma F_i x_i y_i$$

are generally useful,

Let,

$$\Sigma F_i x_i^2 = r^2 \Sigma F_i$$

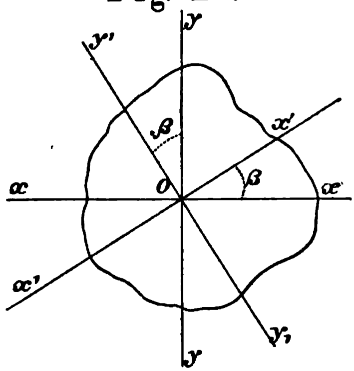
then,

$$r^2 = \frac{\Sigma F_i x_i^2}{\Sigma F_i},$$

where, r, is a length termed the radius of gyration of the given system of forces, relatively to the trace of the sectional plane chosen as axis.

Moments of Inertia analytically considered.—Let, any axis, $y\,y$, traverse the centre of gravity of a materialised surface such as that given in, Fig. 116. Let, x, be the perpendicular distance of any point in the surface from the axis of, y. Conceive a particle placed at this point. For clearness of demonstration, suppose the tendency inherent in this particle to turn the body about an axis traversing a point in the line, $y\,y$, perpendicularly to the plane of the paper, to be in propor-

Fig. 116.



tion to its volume and abscissa multiplied together; or to, $x\,d\,x\,d\,y$. The moment of such an elemental force, relatively to the axis of, y, will be represented by

$$x.\,dx\,dy \times x,$$

and the sum of such elemental moments, taken within the limits of the surface, will be,

$$\iint x^2\,dx\,dy = I \qquad \text{(See Note*)}$$

This expression is termed the moment of inertia of the materialised surface with respect to the axis of, y. In like

* Here the values of x are not taken *absolutely* as in the graphic treatment [§ 7, p. 188], but with regard to their signs, which are used to implicitly distinguish the opposite rotary effects of forces lying on the right and left of the neutral axis, $y\,y$. The same distinction was made in § 7, but in a more direct manner; a *plus* or *minus* sign being attached to each force according to the sense of rotation it tended to produce.

manner the moment of inertia relatively to the axis of, x, will be,

$$\iint y^2\, dx\, dy = J.$$

A third form,

$$\iint x\, y\, dx\, dy = K,$$

expresses the moment about the axis of, x, of forces varying as the product, $xdxdy$.

Let a second system of rectangular axes, $x'x'$, $y'y'$, Fig. 116, traversing as before the centre of gravity of the system, make an angle, $y\hat{O}y' = \beta$, with the first pair. Let the moments of inertia relatively to the new axes be expressed in the following forms:—

$$I_1 = \iint x'^2.\, dx' dy',$$

$$J_1 = \iint y'^2.\, dx'\, dy',$$

$$K_1 = \iint x'.\, y'\, dx'\, dy'.$$

The relations, connecting the new co-ordinates with the old, are, as proved in co-ordinate geometry,

$$\begin{aligned} x' &= x\ cos.\ \beta - y.\ sin.\ \beta. \\ y' &= x\ sin.\ \beta + y.\ cos.\ \beta. \\ x'^2 + y'^2 &= x^2 + y^2 = r^2. \end{aligned}$$

Substituting these values in the expressions given for the new moments of inertia, and remembering that, $dx'\, dy'$, is equal, absolutely speaking, to $dx\, dy$, there results,

$$\begin{aligned} I_1 &= I\ cos.^2\ \beta + J\ sin.^2\ \beta - 2\ K\ cos.\ \beta\ sin.\ \beta, \\ J_1 &= I\ sin.^2\ \beta + J\ cos.^2\ \beta + 2\ K\ cos.\ \beta\ sin.\ \beta, \\ K_1 &= [I - J]\ cos.\ \beta\ sin.\ \beta + K\ [cos.^2 \beta - sin.^2\ \beta]. \end{aligned}$$

Since,

$$x^2 + y^2 = x'^2 + y'^2 = r^2,$$
$$I + J = I_1 + J_1 = \iint (x^2 + y^2)\, dxdy,$$

which is termed the *polar* moment of inertia.

It will also be seen that

$$IJ - K^2 = I_1 J_1 - K_1^2,$$

wherefore, since the polar moment of inertia, $\overline{I_1 + J_1}$, bears a constant value, whilst its component terms, I_1 and J_1, vary with the inclination, β, of the axes; it follows that, I_1, will be a maximum and J_1, a minimum simultaneously. But the maximum value of, I_1, and the minimum value of, J_1, will correspond to the maximum value of, $I_1 - J_1$. Now,

$$[I_1 - J_1]^2 = [I_1 + J_1]^2 - 4 I_1 J_1;$$

hence, the maximum of $[I_1 - J_1]$ corresponds to the minimum of, $I_1 J_1$. Further, it has already been shewn that the quantity, $I_1 J_1 - K_1^2$, has a constant value for all positions of the rectangular system of axes; wherefore the minimum of, $I_1 J_1$, will correspond to the zero-value of, K_1.

Hence, when $K_1 = 0$,

$$I_1 J_1, \text{ is a minimum,}$$
$$I_1 - J_1, \text{ is a maximum,}$$

I_1, a maximum and, J_1, a minimum.

In every plane surface there exist two rectangular axes, traversing the centre of gravity of the surface, for one of which the moment of inertia has a maximum value, I_1, and for the other a minimum value, J_1. These axes are called the principal axes. When these two axes are chosen, from which to calculate the moments of inertia of the surface, the mixed moment, K_1, vanishes. Suppose, then, that the positions of these two axes are unknown. Assume any rectangular axes, $x\,x$, $y\,y$, traversing the centre of gravity of

the surface. Let, I, J, K, represent the moments of inertia relatively to these axes. Take, I_o, J_o, K_o, to represent the moments of inertia relatively to the two principal axes, the position of which is sought; and let, β_o, equal the angle, through which the assumed axes must be turned, in order to coincide with the required principal axes.

By equation [page 194],

$$K_o = [I - J] \cos. \beta_o \sin. \beta_o + K [\cos.^2 \beta_o - \sin.^2 \beta_o].$$

But, according to what has just been shewn, the value of, K_o, vanishes in the case of the principal axes; hence

$$[I - J] \cos. \beta_o \sin. \beta_o + K [\cos.^2 \beta_o - \sin.^2 \beta_o] = 0\,;$$

or,

$$\frac{2 \cos. \beta_o \sin. \beta_o}{\cos.^2 \beta_o - \sin.^2 \beta_o} = \frac{\sin. 2 \beta_o}{\cos. 2 \beta_o} = \tan. 2 \beta_o = \frac{-2K}{I - J}$$

Again, from the equations,

$$I_o + J_o = I + J$$

and

$$I_o J_o - K_o^2 = I J - K^2,$$

are obtained the relations, expressing the moments,

$$I_o = \frac{I + J}{2} + \sqrt{\frac{(I - J)^2}{4} + K^2}$$

$$J_o = \frac{I + J}{2} - \sqrt{\frac{(I - J)^2}{4} + K^2}.$$

In this way are determined the values of the principal moments of inertia, I_o and J_o, as well as the position of the axes to which they are referred.

Having once established the position of the principal axes in the foregoing manner, the moments of inertia, I_2, J_2, K_2, relatively to any other rectangular system, inclined at an

angle, β_2, can be derived from equations [page 194] on the supposition that, $K = K_0 =$ zero ; so that

$$\begin{aligned} I_2 &= I_0 \cos.^2 \beta_2 + J_0 \sin.^2 \beta_2 \\ J_2 &= I_0 \sin.^2 \beta_2 + J_0 \cos.^2 \beta_2 \\ K_2 &= [I_0 - J_0] \sin. \beta_2 \cos. \beta_2 \end{aligned}$$

The method of finding the centre of stress in a given figure subjected to stress uniformly varied, has already been discussed [Pt. III. Ch. I. § 5]. For example, if the stress vary uniformly as the abscissa of its point of application, the resultant stress applied over the whole figure will be expressed by

$$a \iint x \, dx \, dy \, .$$

The sum of the moments of stress about the axis of x, or, in more correct terms, about any axis traversing a point situate on the axis of x, will have the absolute value

$$M_x = a \iint x \, . \, y \, . \, dx \, dy = a \, . \, K \, .$$

and the sum of similar moments about the axis of, y, the value,

$$M_y = a \iint x^2 \, dx \, dy = a \, . \, I.$$

If, therefore, the co-ordinates of the centre of stress, of the figure, be named, x_0 and y_0, we shall have the relations,

$$x_0 = \frac{M_y}{a \iint x \, dx \, dy} = \frac{a . I}{P}$$

$$y_0 = \frac{M_x}{a \iint x \, dx \, dy} = \frac{a \, K}{P} .$$

If, moreover, θ, be the angle contained between the axis of,

y, and the line joining the centre of stress with the origin of co-ordinates,

$$cot.\ \theta = \frac{y_o}{x_o} = \frac{K}{I} \ .$$

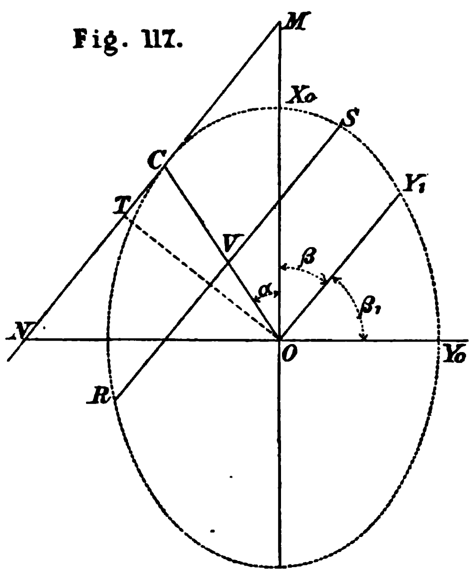
This line, inclined at the angle, θ, to the axis of y, is called the *axis conjugate* of, y.

It has been shewn above [p. 195] that in the case of the principal axes of inertia the value, K, vanishes; hence, in this instance

$$cot.\ \theta = \infty\ ;\ or,\ \theta = 90°.$$

Consequently, the centre of stress will then lie on the axis of, x.

Fig. 117.



Let the moments of inertia relatively to any system of rectangular axis be, I, J, and K, and let it be required to find the moments of inertia, I_1, J_1, and K_1, with respect to a new rectangular system, having for axis of, y, the axis conjugate of, y, in the former system.

The angle, θ, between the new and original axes of, y, will be determined by the relation

$$cot.\ \theta = \frac{K}{I};$$

wherefore,

$$cos.\ \theta = \frac{K}{\sqrt{I^2 + K^2}}.$$

and,

$$sin.\ \theta = \frac{I}{\sqrt{I^2 + K^2}}$$

Substituting these values of, $cos.\ \theta$ and $sin.\ \theta$, in place of $cos.\ \beta$ and $sin.\ \beta$ [equations, page 194], there results,

$$I_1 = \frac{I.[I J - K^2]}{I^2 + K^2};\ \ K_1 = \frac{-K.[I J - K^2]}{I^2 + K^2}.$$

Now, let it be required to find the angle, θ_1, contained between the axis conjugate of, y_1, and the axis of, y_1, itself. By the usual formula,

$$cot.\ \theta_1 = \frac{K_1}{I_1} = \frac{-K}{I} = -\ cot.\ \theta.$$

This relation proves that the axis, y, of the first system is the axis conjugate of the axis, y_1, of the second system of rectangular co-ordinates. Hence, it may be inferred that conjugate axes are mutually conjugate.

Set off, Fig. 117, along the principal axes, x and y, the distances,

$$\overline{O X_o} = \sqrt{I_o} = a,$$
$$\overline{O Y_o} = \sqrt{J_o} = b,$$

and with these two distances as major and minor axes describe an ellipse.

Let the diameter, $\overline{O Y_1}$, be drawn in the direction of any secondary neutral axis, making the angle, $Y_1 \hat{O} Y_o = \beta_1$, with the principal neutral axis, $\overline{O Y_o}$.

Draw the tangent, $\overline{MN}$, parallel to $\overline{OY_1}$, meeting the ellipse of inertia at a point, C. Join, $\overline{OC}$, and let fall the normal, $\overline{OT}$, upon the tangent, $\overline{MN}$. The equation to the line, $\overline{MN}$, is

$$\frac{x}{OM} + \frac{y}{ON} = 1,$$

or, multiplying both sides by, $\overline{OT}$,

$$\frac{OT}{OM} \cdot x + \frac{OT}{ON} \cdot y = \overline{OT}.$$

But,

$$\frac{OT}{OM} = cos. \beta_1 ; \frac{OT}{ON} = sin. \beta_1,$$

whence,

$$x. cos. \beta_1 + y\, sin. \beta_1 = \overline{OT} = n$$

that is

$$y = -\frac{cos. \beta_1 x}{sin. \beta_1} + \frac{n}{sin. \beta_1}.$$

In order that this line may touch the ellipse, at the point, C, its co-ordinates, referred to this point, must satisfy the general equation of the tangent to the ellipse,

$$y = mx \pm \sqrt{a^2 m^2 + b^2}$$

In this instance, $m = \frac{- cos. \beta_1}{sin. \beta_1}$. Hence the two equations will be identical, if

$$\frac{n^2}{sin.^2 \beta_1} = a^2 . \frac{cos.^2 \beta_1}{sin.^2 \beta_1} + b^2,$$

or if,

$$n^2 = a^2 . cos.^2 \beta_1 + b^2 . sin.^2 \beta_1.$$

Let the line, $\overline{OY_1}$, make an angle, β, with the principal axis of, x, and let a diametral line, $\overline{OC}$, bisecting chords parallel to, $\overline{OY_1}$, form an angle, α, with the same axis. Suppose the

line, $\overline{RS}$, to be one of the parallel chords, and x_1, y_1, the co-ordinates of its middle-point, V. Let x, y, be the co-ordinates of, S, and, r, the length of the half-chord, $\overline{VS}$; then,

$$x = x_1 + r\,cos.\,\beta$$
$$y = r\,sin.\,\beta - y_1$$

These values of, x and y, must satisfy the general equation of the ellipse,

$$a^2 y^2 + b^2 x^2 = a^2 b^2$$

that is,

$$a^2 [r\,sin.\,\beta - y_1]^2 + b^2 [x_1 + r.\,cos.\,\beta]^2 = a^2 b^2$$

or, arranging according to the powers of, r,

$$r^2 [a^2 sin.^2 \beta + b^2 cos.^2 \beta] + 2\,r\,[b^2 x_1 cos.\,\beta - a^2 y_1 sin.\,\beta] +$$
$$+ a^2 y_1^2 + b^2 x_1^2 - a^2 b^2 = o$$

This quadratic will give two values for, r, corresponding to, $\overline{VS}$ and $\overline{VR}$. Now, since the chords, $\overline{RS}$, are bisected by the line, $\overline{OC}$, these two values of, r, must be equal in magnitude, but opposite in sign. Hence, by the theory of equations, the coefficient of, r, in the above equation, must vanish, which leads to the relation,

$$b^2 x_1 cos.\,\beta - a^2 y_1 sin.\,\beta = o$$

or,

$$y_1 = \frac{b^2}{a^2}.\,cot.\,\beta.\,x_1$$

Considering the co-ordinates, x_1 and y_1, variable, this is the equation to a straight line passing through the centre of the ellipse, or to the diametral line. $\overline{CO}$, which bisects the series of parallel chords.

Let, a, equal the angle, COX_o; then by the equation of the line, $\overline{CO}$,

$$\frac{y_1}{x_1} = \frac{b^2}{a^2}.\,cot.\,\beta = tan.\,a$$

or,

$$tan.\ \alpha.\ tan.\ \beta = \frac{b^2}{a^2}.$$

But,

$$\beta = [90° - \beta_1],$$

wherefore,

$$tan.^2\ \alpha = tan.^2\ \beta_1 \cdot \frac{b^4}{a^4}$$

Again,

$$\overline{CO}^2 = \frac{a^2 b^2}{a^2 sin.^2 \alpha + b^2 cos.^2 \alpha} = \frac{a^2 b^2}{(a^2 - b^2)\ sin.^2 \alpha + b^2}$$
$$= \frac{a^4 + b^4.\ tan.^2 \beta_1}{a^2 + b^2.\ tan.^2 \beta_1} = \frac{cos.^2 \beta_1\ [a^4 + b^4\ tan.^2 \beta_1]}{n^2}$$

whence,

$$n^2.\ t^2 = n^2.\ \overline{CT}^2 = n^2\ [\overline{CO}^2 - n^2] =$$
$$= a^4.\ cos.^2 \beta_1 + b^4.\ sin.^2 \beta_1 - n^4.$$

Reducing by means of the formula,

$$a^4\ [cos.^2 \beta_1 - cos.^4 \beta_1] = a^4\ cos.^2 \beta_1\ [1 - cos.^2 \beta_1]$$
$$= a^4\ cos.^2 \beta_1.\ sin.^2 \beta_1,$$

there results,

$$n.\ t = [a^2 - b^2].\ cos.\ \beta_1\ sin.\ \beta_1.$$

Comparing the derived equations,

$$n^2 = a^2\ cos.^2 \beta_1 + b^2\ sin.^2 \beta_1,$$
$$n.\ t = (a^2 - b^2)\ cos.\ \beta_1\ sin.\ \beta_1$$

with those deduced for the present case from the general forms given at page 197; namely,

$$I_1 = I_0\ cos.^2 \beta_1 + J_0\ sin.^2 \beta_1$$
$$J_1 = I_0\ sin.^2 \beta_1 + J_0\ cos.^2 \beta_1$$
$$K_1 = [I_0 - J_0]\ cos.\ \beta_1\ sin.\ \beta_1,$$

it becomes apparent that

$I_1 = n^2 = \overline{OT}^2 =$ moment of inertia about, $\overline{OY_1}$
$K_1 = n.\ t = \overline{CT} \times \overline{OT},$

and

$$cot.\, \theta = \frac{K_1}{I_1} = \frac{t}{n} = \frac{\overline{CT}}{\overline{OT}} = cot.\, Y_1\hat{O}C,$$

so that the conjugate axis of stress, inclined at an angle, $\theta = Y_1OC$, to the neutral axis of, y, coincides with the elliptic conjugate axis, $\overline{CO}$, of the semi-diameter, $\overline{OY_1}$.

When the given plane surface, of which it is required to find the moment of inertia, is a very complex figure, it may be more convenient in certain cases to find the moments of inertia of separate parts of the surface, each with respect to a neutral axis traversing the centre of gravity of that part. And, as in most cases, these different neutral axes will not traverse the centre of gravity of the whole surface, it becomes necessary to reduce these several independent moments of inertia, to an axis traversing the general centre of gravity.

Suppose, then, x_1, be the distance of a point, P, Fig, 118, from the independent neutral axis, $\overline{Y_1 Y_1}$, situate in one of the parts and traversing its centre of gravity. The moment of inertia of the part, relatively to the axis, $\overline{Y_1 Y_1}$, will be

$$\iint x^2_1\, dx\, dy.$$

Now let the distance of the point, P, from the general neutral axis, $\overline{Y_\circ Y_\circ}$, be represented by, x;—then, if, $r_\circ$, be the distance between, $\overline{Y_\circ Y_\circ}$ and $\overline{Y_1 Y_1}$,

$$x = r_\circ + x_1$$
$$x^2 = r_\circ^2 + x_1^2 + 2\, r_\circ\, x_1,$$

and the reduced moment of inertia of the part considered relatively to the axis, $\overline{Y_\circ Y_\circ}$, will be,

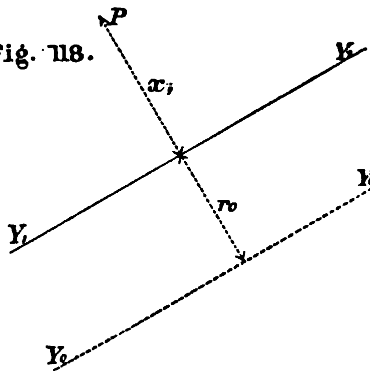
$$\iint x^2\, dx\, dy = \iint [r_\circ^2 + x_1^2 + 2r_\circ\, x_1]\, dx\, dy.$$
$$= r_\circ^2 \iint dx\, dy + 2r_\circ \iint x_1\, dx\, dy + \iint x^2_1\, dx\, dy.$$

But, since, $\overline{Y_1\,Y_1}$, traverses the centre of gravity of the part-area, the integral, $\iint x_1\,dx\,dy$, will vanish; hence, finally

$$\iint x^2\,dx\,dy = \iint x_1^2\,dx\,dy + r_\circ^2 \iint dx\,dy,$$

which may be interpreted to mean that the reduced moment of inertia of a part of a surface, about an axis traversing the centre of gravity of the whole surface, is equal to the moment

Fig. 118.



of inertia of the same part about a neutral axis traversing its own centre of gravity, parallel to the general axis;—*plus* the product of the area of the part, and the square of the distance of its centre of gravity from the general neutral axis.

Moments of Inertia may be treated analytically in another way. Let a point, B, Fig. 119, be the point of application of a force, F_i, forming part of any given system. Suppose, $\overline{B\,A} = p$, to be the perpendicular distance of the point, B, from the line, $\overline{O\,T}$.

The moment of inertia of the given force, F_i, relatively to the line, $\overline{O\,T}$, will be

$$\begin{aligned} F_i \times \overline{B\,A}^2 &= F_i\,.\,[B\,D - A\,D]^2 \\ &= F_i.\,[B\,D - E\,K]^2 = F_i.\,[y_i\,cos.\,a - x_i\,sin.\,a]^2. \end{aligned}$$

The moment of inertia of the series of forces belonging to the same system will be

$$\Sigma\,[F_i.\,(y_i\,cos.\,a - x_i\,sin.\,a)^2]$$

Hence, if, r, represent the radius of gyration of the system relatively to the line, $\overline{OT}$, we have,

$$r^2 . \Sigma . F_i = \Sigma \left[F_i . (y_i \cos . a - x_i \sin . a)^2\right]$$

or,

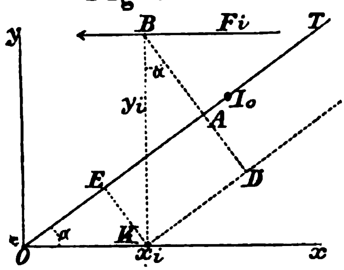
$$r^2 \Sigma . F_i = \Sigma . F_i . y_i^2 \cos .^2 a + . \Sigma . F_i . x_i^2 \sin .^2 a - 2 \Sigma F_i x_i y_i \sin . a \cos . a. \qquad (1)$$

Let,

$$a^2 \Sigma . F_i = \Sigma . F_i . x_i^2$$
$$b^2 \Sigma . F_i = \Sigma . F_i y_i^2,$$

in which equations, a, represents the radius of gyration of the system of forces relatively to the axis of, y; and, b, that with

Fig. 119



respect to the axis of, x. If, by way of analogy, it be agreed to make

$$c^2 \Sigma . F_i = \Sigma . F_i x_i y_i,$$

the equation given above (1) takes the form,

$$r^2 = b^2 \cos .^2 a + a^2 \sin .^2 a - 2 c^2 \sin . a \cos . a \qquad (2)$$

[Compare value of I_1. p. 194].

Along the line, $\overline{OT}$, set off a distance, $\overline{OI_o}$, equal to, $\frac{\mu}{r}$; where, μ, is an arbitrary constant. The co-ordinates of the point, I_o, will consequently be

$$x_o = \overline{OI_o} \cos . a ; \; y_o = \overline{OI_o} \sin . a$$

or, since $O\,I_o = \frac{\mu}{r}$

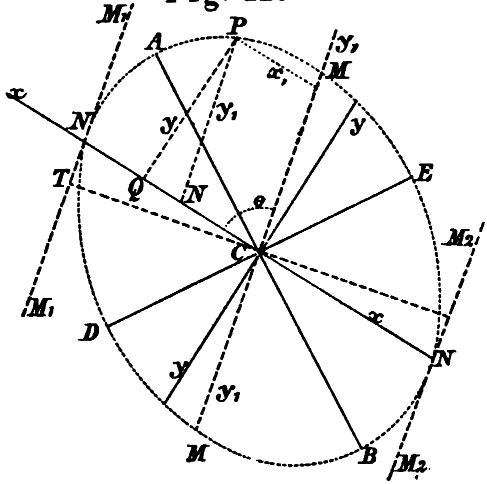
$$x_o = \frac{\mu}{r}.\cos.\,a\,;\ y_o = \frac{\mu}{r}\sin.\,a\,;$$

whence,

$$\cos.\,a = \frac{r}{\mu}.\,x_o\,;\ \sin.\,a = \frac{r}{\mu}.\,y_o$$

Substituting the above values of $\cos.\,a$ and $\sin.\,a$ in the equation of the radius of gyration, [eq, 2., p. 205], we obtain

Fig. 120



the general equation to the locus of points, such as, I_o, namely,

$$r^2 = b^2.\frac{r^2}{\mu^2}.\,x_o^{\ 2} + a^2.\frac{r^2}{\mu^2}.\,y_o^{\ 2} - 2\,c^2\frac{r^2}{\mu^2}.\,x_o\,y_o,$$

or,

$$\mu^2 = b^2\,x^2 + a^2\,y^2 - 2\,c^2\,x\,y. \qquad (3).$$

If the rectangular axes be so chosen as to coincide with the principal axes of inertia [pp. 194-5] the term

$$c^2 = \frac{\Sigma.\,F_i\,x_i\,y_i}{\Sigma.\,F_i} = \frac{K,}{\Sigma.\,F_i},$$

vanishes. Hence, the general equation to the locus of

points, I_o, referred to the principal axes of inertia takes the form,

$$b^2 x^2 + a^2 . y^2 = \mu^2.$$

Now, μ^2, has all along been considered arbitrary. Make it equal to, $a^2 b^2$; then

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

in which, by definition, a and b, represent the radii of gyration relatively to the principal axes of inertia, y and x.

It has already been remarked that the moment of inertia of a force takes the same sign as the force; so that in certain cases, where the forces are of opposite sign, the resultant moments of inertia, $\Sigma . F_i x_i^2$ and $\Sigma . F_i y_i^2$, may differ in sign; whilst the sum, $\Sigma . F_i$, common to the values, a^2 and b^2, has necessarily the same sign in both expressions. The moments may differ in sign, because they depend for their absolute magnitudes on the squares of the distances of the points of application from the axes of, x and y, and for their sign only upon the forces themselves. Wherefore, the values,

$$a^2 = \frac{\Sigma . F_i x_i^2}{\Sigma . F_i}; \; b^2 = \frac{\Sigma . F_i y_i^2}{\Sigma . F_i},$$

may differ in sign, and the general equation to the locus of the points, I_o, namely,

$$a^2 y^2 + b^2 x^2 = a^2 b^2$$

may represent an hyperbola instead of an ellipse.

Choose two axes, Cx, Cy, Fig. 120, at right angles to each other;—the equation of the ellipse of inertia, referred to, C, as a centre and to, Cx, Cy, as axes of co-ordinates will be, [eq. 3, p. 206]

$$b^2 x^2 + a^2 y^2 - 2 c^2 x y = \mu^2,$$

in which,

$$b^2 = \frac{\Sigma . F_i y_i^2}{\Sigma . F_i}; \; a^2 = \frac{\Sigma . F_i x_i^2}{\Sigma . F_i}$$

$$c^2 = \frac{\Sigma . F_i x_i y}{\Sigma . F_i},$$

and, μ, is an arbitrary constant.

Now it has been previously shewn [eq. (2). p. 205] that if, r, be the radius of gyration relatively to any line, $\overline{CM}$, inclined at an angle, θ, to the axis of, x; then

$$r^2 = b^2 \cos.^2 \theta + a^2 \sin.^2 \theta - 2 c^2 \sin.\theta \cos.\theta.$$

Let us take the line, $\overline{CM}$, as a new axis of, y_1. We shall, in that case, have the following relations obtaining between the new and old co-ordinates of the same point,

$$y = y_1 \sin.\theta \,; \; x = x_1 + y_1 \cos.\theta.$$

Consequently equation 3, page 206, referred to the new axes, will take the form,

$$b^2 [x_1 + y_1 \cos.\theta]^2 + a^2 . y_1^2 \sin.^2 \theta - 2c^2 . y_1 \sin.\theta [x_1 + y_1 \cos.\theta] = \mu^2$$

or, more symmetrically,

$$b^2 x_1^2 + [2 b^2 \cos.\theta - 2 c^2 \sin.\theta] x_1 y_1 + [b^2 \cos.^2 \theta + a^2 \sin.^2 \theta - \\ - 2 c^2 \sin.\theta \cos.\theta] y_1^2 = \mu^2.$$

But the coefficient of, y_1^2, is equal to, r^2, [see above], or to the radius of gyration relatively to the axis, $\overline{CM}$. If, for the sake of symmetry, r^2, be denoted by, a_1^2; so that

$$a_1^2 = \frac{\Sigma. F_i {x^1}_i^2}{\Sigma. F_i}$$

and

$$c_1^2 = \frac{\Sigma. F_i x_i^1 y_i^1}{\Sigma. F_i},$$

the equation to the ellipse of inertia, referred to the new axes, can be deduced in the following manner:—

By relations already established,

$$x_1 = x - y_1 \cos.\theta = x - y . \frac{\cos.\theta}{\sin.\theta},$$

$$y_1 = \frac{y}{\sin.\theta};$$

whence,

$$x_1 y_1 = \frac{x y}{sin. \theta} - y^2 . \frac{cos. \theta,}{sin.^2 \theta}$$

or more generally for any point, $x_i^1 y_i^1$,

$$x_i^1 y_i^1 = \frac{x_i y_i}{sin. \theta} - y_i^2 \frac{cos. \theta.}{sin^2. \theta}$$

Wherefore,

$$\frac{\Sigma. F_i x_i^1 y_i^1}{\Sigma. F_i} = \frac{1}{sin. \theta} . \frac{\Sigma. F_i x_i y_i}{\Sigma. F_i} - \frac{cos. \theta}{sin.^2 \theta} . \frac{\Sigma. F_i y_i^2}{\Sigma. F_i} ;$$

that is,

$$c_1^2 = \frac{1}{sin. \theta} . c^2 - \frac{cos. \theta.}{sin.^2 \theta} b^2$$

or

$$c_1^2 \, sin.^2 \theta = c^2 \, sin. \theta - b^2 \, cos. \theta,$$

from which can be deduced the relation,

$$2 [b^2 cos. \theta - c^2 sin. \theta] = - 2 c_1^2 sin.^2 \theta.$$

Substituting in equation [p. 208] there results

$$b^2 x_1^2 + a_1^2 y_1^2 - 2 c_1^2 sin.^2 \theta \, x_1 y_1 = \mu^2,$$

which is the equation of the ellipse referred to the new axes. Since the axis of, x, is common to both systems of axes, the radius of gyration, b, does not suffer any change.

Suppose that the new axes, Cx and $\overline{CM}$, are conjugate diameters of the ellipse, and for symmetry, let

$$b = b_1 ; \ \mu^2 = a_1^2 b_1^2 ;$$

then, since the term in, $x_1 y_1$, vanishes, when the ellipse is referred to its conjugate diameters,

$$c_1^2 = \frac{\Sigma. F_i x_i^1 y_i^1}{\Sigma. F_i} = o,$$

and the transformed equation takes the simpler form,

$$b_1^2 x_1^2 + a_1^2 y_1^2 = a_1^2 b_1^2. \qquad (4.)$$

The terms, a_1 and b_1, may be called oblique radii of gyration.

The ellipse of inertia can be also defined by means of line-envelopes; for, if through the centre, C, a series of lines, $\overline{MM}$, be drawn;—and parallel and corresponding to this series, lines, $\overline{M_1M_1}$, $\overline{M_2M_2}$, be set off, distant from the centre, C, to which they refer, by a length equal to the radius of gyration taken relatively to the lines, $\overline{MM}$;—the envelope of these lines, or the curve formed by their continuous intersections will determine the ellipse of inertia. For, by a former demonstration, if, r, be the radius of gyration with respect to, $\overline{MM}$, [see p. 206]

$$\overline{CM} = \frac{\mu}{r},$$

from which,

$$r = \frac{\mu}{CM} = \frac{\mu}{b_1} = \frac{a\,b}{b_1}$$

Now, since the line, $\overline{CM}$, is parallel to the line, $\overline{M_1M_1}$, drawn tangent to the ellipse, at the extremity of the diameter, $\overline{NN}$,

$$\overline{CN} = a_1.$$

By a well-known theorem,

$$a\,b = a_1\,b_1\,sin.\,\theta\,;$$

hence,

$$r = \frac{a\,b}{b_1} = \frac{a_1\,b_1\,sin.\,\theta}{b_1} = a_1\,sin.\,\theta = \overline{CT}.$$

It may be, therefore, inferred that the series, $\overline{M_1M_1}$ and $\overline{M_2M_2}$, will envelop the contour of the ellipse of inertia.

The normal, $\overline{CT}$, which here represents the radius of gyration, can be used equally as well to graphically depict the moment of inertia [see p. 202]. It is only necessary to multiply it by the constant factor, $\sqrt{\Sigma.\,F_i}$, and square the product; thus

$$r.^2\,\Sigma.\,F_i = \Sigma.\,F_i\,x_i^2$$

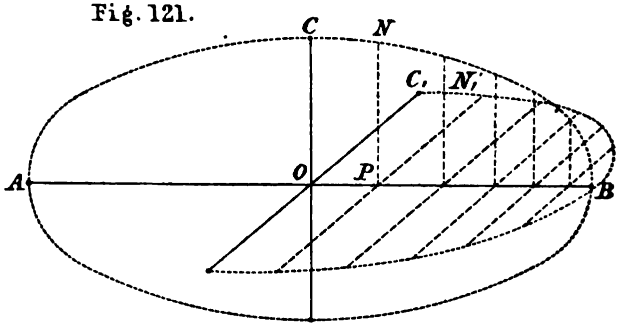
$$r = \sqrt{\frac{\Sigma.\,F_i\,x_i^2}{\Sigma.\,F_i}} = \sqrt{\frac{I}{\Sigma.\,F_i}}$$

It will be observed that the abscissæ, x_i, are measured at right angles to the diameter, $\overline{MM}$, and not parallel to the conjugate diameter, $\overline{NN}$, as was done in establishing the relation, [(4) p. 209], in which

$$a_1^2 \Sigma . F_i = \Sigma . F_i \, (x_i^1)^2.$$
$$b_1^2 \Sigma . F_i = \Sigma . F_i \, (y_i^1)^2.$$

Let there be given two conjugate semi-diameters, $\overline{OC_1}$ and $\overline{OB}$, Fig. 121, and let it be required to construct the ellipse to which they belong.

Fig. 121.



Set off, $\overline{OC}$, at right angles to, $\overline{OB}$, and equal in length to the conjugate axis, $\overline{OC_1}$. Next, construct an ellipse upon, $\overline{OC}$ and $\overline{OB}$, as principal axes. In this auxiliary ellipse draw any ordinate, $\overline{PN}$, and from the point, P, draw a second ordinate, $\overline{PN_1}$, parallel to, $\overline{OC_1}$, and equal in length to, $\overline{PN}$. The extremities, N_1, of a series of ordinates, drawn according to the same principle, will determine the required ellipse. For, let

$$\overline{OB} = a_1 ; \ \overline{OC_1} = b_1 ;$$

then the equation to the ellipse sought will be,

$$a_1^2 y_1^2 + b_1 x_1^2 = a_1^2 b_1^2,$$

which can be put in the form,

$$\frac{y_1^2}{(a_1 + x_1)(a_1 - x_1)} = \frac{b_1^2}{a_1^2} ;$$

or again, in the geometric form,

$$\frac{\overline{PN_1}^2}{\overline{AP} \times \overline{BP}} = \frac{b_1^2}{a_1^2} = \text{a constant.}$$

If now the equation of the auxiliary ellipse be stated, a similar relation exists for the point, P;—namely,

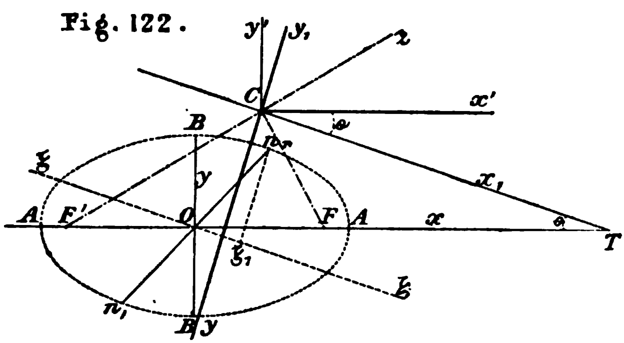
$$\frac{\overline{PN}^2}{\overline{AP} \times \overline{BP}} = \frac{b_1^2}{a_1^2} = \text{a constant};$$

from which it immediately follows that

$$\overline{PN} = \overline{PN_1}.$$

If the central ellipse of inertia of a system be known; that is, the ellipse of inertia referred to the principal axes, traversing the centre of forces of the given system;—it may be further required to find the direction of the principal axes of a second ellipse of inertia, relatively to another point, situate in the same plane as the central ellipse. For example, suppose the given central ellipse, $A\,B\,A\,B$, Fig. 122, to lie in the same

Fig. 122.



plane with the point, C, about which it is sought to construct the corresponding ellipse of inertia.

Assume the principal axes of the central ellipse as axes of co-ordinates, the origin being chosen at the centre, O;—and let the co-ordinates of the fixed point, C, be, x_o, y_o.

Take the square of the focal distance,

$$\overline{OF}^2 = -\overline{OF'}^2 = \overline{OA}^2 - \overline{OB}^2 = a^2 - b^2 = -k^2,$$

where, according to the nature of the ellipse of inertia,

$$a^2 = \frac{\Sigma.\, F_i\, x_i^2}{\Sigma.\, F_i}\,;\; b^2 = \frac{\Sigma.\, F_i\, y_i^2}{\Sigma.\, F_i}.$$

Through the point, C, draw a pair of rectangular axes, $x'\,y'$, respectively parallel to x and y, and let, a' and b', express the radii of gyration corresponding to the new system; so that

$$a'^2 = a^2 + x_o^2\,;\; b'^2 = b^2 + y_o^2,$$

Let, $x_i'\,y_i'$, be the co-ordinates of the point of application of any force with respect to the axes, $x'\,y'$, and suppose

$$c'^2 = \frac{\Sigma.\, F_i\, x'_i\, y_i'}{\Sigma.\, F_i}.$$

The co-ordinates, $x_i\,y_i$, taken with respect to the original axes, may be expressed in the form,

$$x_i = x_o + x'_i\,;\; y = y_o + y'_i.$$

Hence,

$$\begin{aligned} x'_i\,y'_i &= (x_i - x_o)\,(y_i - y_o) \\ &= x_i\,y_i - x_i\,y_o - x_o\,y_i + x_o\,y_o\,; \end{aligned}$$

Consequently,

$$\begin{aligned} c'^2 &= \frac{\Sigma.\, F_i\, x'_i\, y'_i}{\Sigma.\, F_i} = \\ &= \frac{\Sigma.\, F_i\, x_i\, y_i}{\Sigma.\, F_i} - \frac{\Sigma.\, F_i\, x_i\, y_o}{\Sigma.\, F_i} - \frac{\Sigma.\, F_i\, x_o\, y_i}{\Sigma.\, F_i} + x_o\, y_o. \end{aligned}$$

But, in the case of the principal axes of inertia, the term, $\Sigma.\, F_i\, x_i\, y_i$, vanishes; [p. 195], so also will the term,

$$\Sigma.\, F_i\, x_i\, y_o = y_o\, \Sigma.\, F_i\, x_i,$$

because the factor, $\Sigma.\, F_i\, x_i$, expresses the sum of the moments of forces about a line traversing the centre of forces. In like manner the third term,

$$\Sigma.\, F_i\, x_o\, y_i = x_o\, \Sigma.\, F_i\, y_i$$

will vanish, leaving the equation in the form,

$$c'^2 = x_o y_o.$$

The equation of the ellipse of inertia with respect to the point, C, will therefore be

$$(a^2 + x_o^2) y'^2 + (b^2 + y_o^2) x'^2 - 2 x_o y_o . x' y' = \mu^2$$

Now, let the unknown direction of the principal axes of the ellipse of inertia, referred to the point, C, form an angle, θ, with the direction of the axes, $x' y'$; and let, x_1, y_1, be the co-ordinates of any point, measured parallel to the required principal axes. We can pass from the system, $x' y'$, to that of, $x_1 y_1$, by means of the usual relations, connecting two rect-angular systems, inclined to each other at an angle, θ;—viz.,

$$x' = x_1 \cos. \theta - y_1 \sin. \theta$$
$$y' = x_1 \sin. \theta + y_1 \cos. \theta$$

Substituting these values of, x' and y', in the above general equation, and stating the equation, peculiar to the ellipse of inertia referred to its principal axes, that the coefficient of, $x_1 y_1$, should vanish, there results the following relation for the determination of the angle, θ,

$$\begin{aligned} \tan. 2\theta &= \frac{2 x_o y_o}{(a^2 + x_o^2) - (b^2 + y_o^2)} \\ &= \frac{2 x_o y_o}{(a^2 - b^2) + x_o^2 - y_o^2} \\ &= \frac{2 x_o y_o}{x_o^2 - k^2 - y_o^2} \\ &= \frac{\dfrac{y_o}{x_o - k} + \dfrac{y_o}{x_o + k}}{1 - \dfrac{y_o}{x_o - k} . \dfrac{y_o}{x_o + k}} \end{aligned}$$

But,

$$\frac{y_o}{x_o - k} = \tan. C\hat{F}x; \quad \frac{y_o}{x_o + k} = \tan. C\hat{F}'x;$$

hence

$$tan.\,[C\hat{F}x + C\hat{F}'x] = \frac{tan.\,C\hat{F}x + tan.\,C\hat{F}'x}{1 - tan.\,C\hat{F}x.\,tan.\,C\hat{F}'x}$$

$$= \frac{\dfrac{y_0}{x_0 - k} + \dfrac{y_0}{x_0 + k}}{1 - \dfrac{y_0}{x_0 - k} \cdot \dfrac{y_0}{x_0 + k}}$$

$$= tan.\,2\,\theta.$$

Wherefore

$$2\,\theta = [C\hat{F}x + C\hat{F}'x]$$

and

$$\theta = \frac{1}{2}\,[C\hat{F}x + C\hat{F}'x]\,;$$

or,

$$\theta = \frac{\pi}{2} - \frac{1}{2}\,[C\hat{F}x + C\hat{F}'x]$$

Produce the focal line, $F'C$, to z; and bisect the angle, $F\hat{C}z$, by the line, $\overline{CT}$;—then,

$$\begin{aligned} F\hat{C}z &= 2.\,z\hat{C}T \\ &= 2\,[\,z\hat{C}x' + x'\hat{C}T\,]. \end{aligned}$$

Again,

$$\begin{aligned} F\hat{C}z &= [C\hat{F}'x + \overline{180^\circ - C\hat{F}x}] \\ &= 2\,[z\hat{C}x' + x'\hat{C}T] \end{aligned}$$

But, $z\hat{C}x' = C\hat{F}'x$; therefore

$$[C\hat{F}'x + \pi - C\hat{F}x] = 2.\,[C\hat{F}'x + x'\hat{C}T]\,;$$

whence by transposition

$$\pi - 2\,x'\hat{C}T = C\hat{F}x + C\hat{F}'x$$

or

$$\frac{\pi}{2} - x'\hat{C}T = \frac{1}{2}\,[C\hat{F}x + C\hat{F}'x]$$

Hence, [see above], in this instance, $\theta = x'\hat{C}T$, and,

$$\frac{\pi}{2} - \theta = \frac{1}{2}\,[C\hat{F}x + C\hat{F}'x]$$

The axis of, y_1, will be perpendicular to that of, x_1, and will consequently bisect the angle, $F\hat{C}F'$, supplement of $z\hat{C}F$.

In order to find the radii of gyration relatively to the principal axis through, C; it is only necessary to draw a line, $\xi_1\xi_1$, passing through, O, and parallel to the axis, $\overline{x_1 x_1}$, as well as the axis, $\overline{n_1 n_1}$, conjugate of, $\xi_1\xi_1$. The perpendicular, $n_1\xi_1$, let fall upon, $\xi_1\xi_1$, from the extremity, n_1, of the conjugate diameter, will represent the radius of gyration relatively to the axis, $\xi_1\xi_1$. [p. 210].

Now, let the perpendicular distance between the axes, $\xi_1\xi_1$ and $x_1 x_1$, be represented by, p; then the radius of gyration relatively to the axis, $x_1 x_1$, will be

$$b_1 = \sqrt{\xi_1 n_1^2 + p^2}$$

It has been previously demonstrated [p. 190] that, by means of the polygon of forces and the polar polygon, the moment of inertia of any system of forces can be determined relatively to any special axis; and since the radius of gyration depends for its expression upon the moment of inertia, it follows that it also can be graphically determined. The truth of this statement has been profusely illustrated in the foregoing pages. But in some cases when the forces are unlimited in number, as for instance in the case of gravity acting on a materialised surface, the methods of the integral calculus are to be preferred.

If, δ, represent the density of an elemental part, $dxdy$, of a plane surface, A, Fig. 123, the moment of inertia of the surface, relatively to the axis of x, will be

$$I = \iint \delta . \, y^2 \, dxdy,$$

and its weight,

$$W = \iint \delta \, dxdy.$$

Consequently, its radius of gyration with respect to the axis of, x, will be

$$r = \sqrt{\frac{I}{W}}. \quad \text{[Compare p. 192].}$$

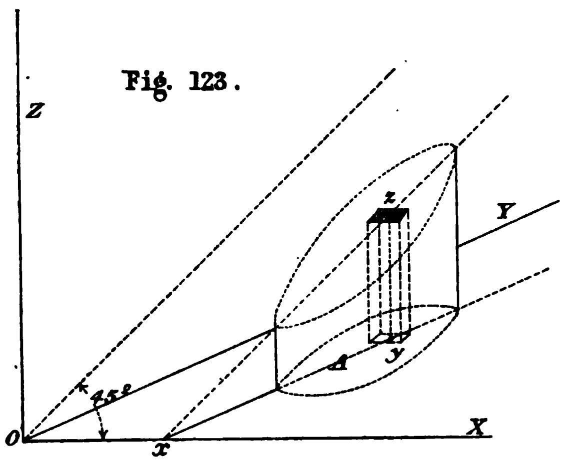
Another process of finding the radius of gyration, depends on the following principle, due originally to Bresse.

Given the plane area, A, choose any axes, x, y, situate in this plane, and let, y_0, be the ordinate of the centre of gravity of the area, A; then, by a well-known principle,

$$y_0 \iint \delta .\, dxdy = \iint \delta . y\, dx\, dy.$$

Upon the base, A, superpose, as shewn, a cylinder with its generating lines normal to the plane, xy. Let a part of this cylinder be cut off by a plane of section, passing through the

Fig. 123.



axis of, x, and inclined to the plane of the base at an angle of 45°. The volume of the intercepted prism, standing upon the elemental area, $dx\, dy$, will be

$$z .\, dx\, dy.$$

But the triangle, $x\, y\, z$, is isosceles, the angles, at, x and z, being each equal to 45 degrees; hence, since $z = y$,

$$z .\, dx\, dy = y\, dx\, dy$$

The weight of this prismatic intercept will therefore be

$$\omega = \delta .\, y\, dx\, dy$$

and its moment relatively to the axis of, x,

$$m = \delta .\ y^2 .\ dx\,dy$$

Consequently, if, y'_o, represent the ordinate of the projection of the centre of gravity of the cylindrical intercept upon the plane of, $x\,y$,

$$y'_o . \iint \delta .\ y\,dx\,dy = \iint \delta .\ y^2\,dx\,dy \qquad (1)$$

But it has just been shewn that, y_o, being the ordinate of the centre of gravity of the base, A,

$$y_o \iint \delta .\ dx\,dy = \iint \delta\ y\,dx\,dy. \qquad (2)$$

Multiplying together equations, (1), and, (2),

$$y'_o . y_o . \iint \delta .\ dx\ dy . \iint \delta .\ y\ dx\ dy =$$

$$= \iint \delta .\ y^2\ dx\ dy . \iint \delta .\ y\,dx\,dy\,;$$

and suppressing the common factor, $\iint \delta .\ y\,d\,x\,dy$,

$$y'_o . \ y_o = \frac{\iint \delta .\ y^2 .\ dx\,dy}{\iint \delta .\ dx\,dy}$$

$$= \frac{I}{W} = r^2\,;$$

therefore,

$$r = \sqrt{y' .\ y_o}$$

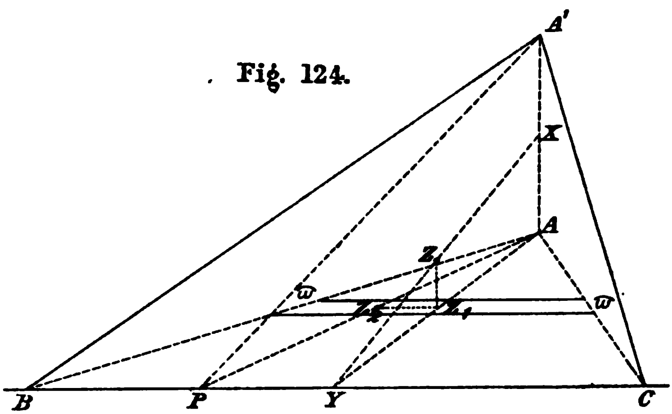
Applying the above principle to the particular case of the triangle, $A\,B\,C$, Fig. 124, let it be required to find the radius of gyration relatively to one of its sides chosen as axis, for example, the side, $\overline{B\,C}$. Suppose, $\overline{A\,P}$, shewn in perspective, to be the perpendicular let fall from the vertex, A, upon the side, $\overline{B\,C}$. The distance from the base, $\overline{B\,C}$, of the centre of gravity of the triangle will be

$$y_o = \frac{A\,P}{3} = \frac{h}{3} \qquad (3)$$

Let a plane passing through the axis of x; that is, through the base, $\overline{BC}$, form an angle of 45° with the plane of, xy, and imagine a right-prism, erected on $\overline{ABC}$, as a base, to be divided by this plane of section. Let, $\overline{AA'}$ represent the line-limit of the prism normal to the plane of the triangle, ABC.

It will be seen by the above construction that the triangle, APA', is isosceles, since the angles at, A' and P, are each equal to 45°.

Fig. 124.



The centre of gravity of the tetrahedron, having its four summits at, B, C, A, and A', is situate at the middle point, Z, of the line, $\overline{XY}$, joining the centre points, X and Y, of the lines, $\overline{AA'}$ and $\overline{BC}$, mutually opposed. The projection of the centre of gravity, Z, upon the plane of, xy, will be found at, Z_1, the extremity of the perpendicular, $\overline{ZZ_1}$, let fall from Z, upon the base-area, ABC. Through, Z_1, draw a line, $\overline{Z_1 Z_2}$, parallel to, $\overline{BC}$, intersecting the line, $\overline{AP}$, in a point, Z_2. Then the distance of the point, Z_1, from the base-line, $\overline{BC}$; or in other terms its ordinate parallel to the axis of, y, will be

$$y'_{\circ} = \overline{PZ_2} = \frac{AP}{2} = \frac{AA'}{2} = \frac{h}{2}. \qquad (4)$$

wherefore, multiplying together equations, (3) and (4)

$$y_{\circ} \times y_{\circ}' = r^2 = \frac{h}{3} \times \frac{h}{2} = \frac{h^2}{6}$$

or,

$$r = \frac{h}{\sqrt{6}}$$

The integral calculus would furnish the same result; for, taking the line, $\overline{BC}$, as the axis of, x, and a line perpendicular to it, as the axis of, y, we find the area of a small elemental strip, $\omega\,\omega$, of the triangle, distant, y, from the axis of x, by means of the following relation,

$$\frac{b}{\overline{BC}} = \frac{h - y.}{h}$$

in which, b, represents the mean breadth of the strip. Hence, the area of the strip can be put into the form,

$$b\,d\,y = \overline{BC}.\,\frac{h - y.}{h}\,dy,$$

and taking the density as constant and equal to unity, the moment of inertia of the triangle with respect to the axis, $\overline{BC}$, can be expressed as,

$$\int_0^h \frac{\overline{BC}}{h}.\,[h - y].\,y^2\,dy = BC\left[\frac{h^3}{3} - \frac{h^3}{4}\right] = \frac{\overline{BC}.h^3}{12}.$$

Consequently its radius of gyration, relatively to the same line, will be given by the relation,

$$r^2 = \overline{BC}.\frac{h^3}{12} \div \overline{BC}.\frac{h}{2} = \frac{h^2}{6};$$

whence, as before,

$$r = \frac{h}{\sqrt{6}}$$

Suppose it were now required to find the moment of inertia of the triangle, $A\,B\,C$, Fig. 125, relatively, not to one of its sides, such as, $\overline{BC}$, but to a line, $B\,x$, passing through the vertex, B, in any direction.

Produce the side, $\overline{AC}$, to D, and find, by the usual rule, the moment of inertia of the triangle, $B\,A\,D$, with respect to the base, $\overline{BD}$, and secondly the moment of inertia of the triangle, $B\,C\,D$, relatively to the same line. Then, the moment of

inertia of the triangle, $A\,B\,C$, with respect to $B\,D$, will be equal to the difference of the two moments just found.

The moment of inertia relatively to a line, $\overline{Z\,Z}$, parallel to, $B\,x$, could be determined either by first finding the similar moment with respect to an axis, parallel to, $\overline{Z\,Z}$, traversing the centre of gravity of the figure; and subsequently taking

Fig. 125.

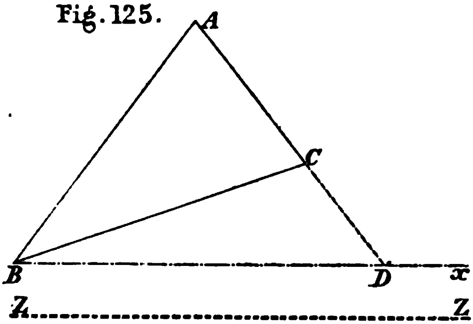
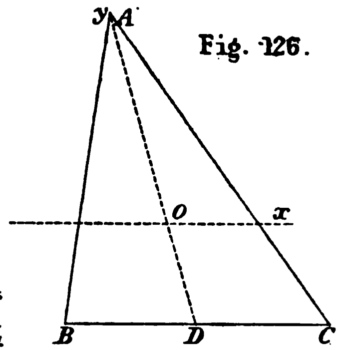


Fig. 126.



the sum of this moment and a term composed of the square of the distance of the axis $\overline{Z\,Z}$ from the centre of gravity of the triangle, $A\,B\,C$, multiplied by the area of the same figure;—or again use might be made of the known moment of inertia relatively to some line, $B\,x$, parallel to, $\overline{Z\,Z}$, in order to find the auxiliary moment about the parallel axis traversing the centre of gravity, which could then be employed as already explained.

Let it be required to find the *central ellipse of inertia* of the triangle, $A\,B\,C$, Fig. 126. This ellipse must have its centre at the centre of gravity of the figure; so that if the line, $\overline{A\,D}$, be drawn from the vertex, A, to the point of bisection, D, of the base-line, $\overline{B\,C}$, the centre of the ellipse will be at a point, O, on this line, where,

$$\overline{O\,D} = \frac{A\,D}{3}$$

Here the line, $\overline{A\,D}$, is a symmetrical centre-line, [Pt. I. Ch. IV. § 3]; that is, correlative or paired forces are disposed, not only at equal distances on each side of it, but can be so arranged, that the forces comprising the pairs are equal in magnitude. By the

article above referred to, it follows that if lines, $\overline{OA}$ and Ox, be taken as co-ordinate axes, the sum, $\Sigma.\ F_i.\ x_i.\ y_i.$, vanishes, that is the chosen axes will be conjugate. In order therefore to construct the central ellipse it is only necessary to find the radii of gyration relatively to the axes, $\overline{OA}$ and Ox. Let, h, be the length of the perpendicular, let fall from, A, on $\overline{BC}$;—then the square of the radius of gyration with respect to the axis, $\overline{BC}$, will be, [p. 219]

$$r^2 = \frac{h^2}{6}.$$

Now, the perpendicular distance of the centre of gravity, O, from the line, $\overline{BC}$, is equal to, $\frac{h}{3}$;—hence the radius of gyration relatively to the axis, Ox, will be

$$r_\circ = \sqrt{r^2 - \left(\frac{h}{3}\right)^2} = \sqrt{\frac{h^2}{18}} = \frac{h}{3\sqrt{2}}.$$

The square of the radius of gyration relatively to the axis, $\overline{OA}$, of the whole triangle will be double that of its half, ADC. Let, k, be the perpendicular distance of the vertex, C, from AD,—the square of the radius of gyration of ADC, with respect to, AD, is, [p. 219.]

$$r_1^2 = \frac{k^2}{6}.$$

Therefore, $r_\circ'$, being the radius of gyration of the whole triangle relatively to the axis, $\overline{OA}$,

$$(r_\circ')^2 = 2\,r_1^2 = \frac{k^2}{3}$$

or

$$r_\circ' = \frac{k}{\sqrt{3}}$$

It is now only necessary to set off along, Ox, a distance, a'; and along, Oy, a distance, b', such that

$$a' = \frac{\mu}{r_\circ} = \mu \div \frac{h}{3\sqrt{2}};\quad b' = \frac{\mu}{r_\circ'} = \mu \div \frac{k}{\sqrt{3}}$$

where, μ, is any arbitrary unit of scale.

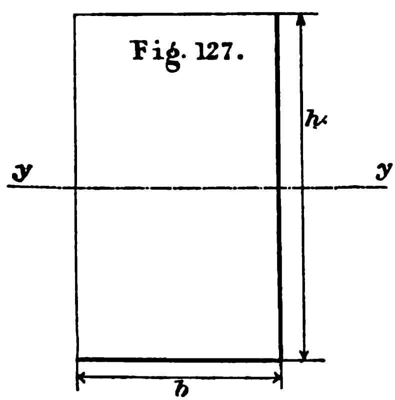
The ellipse of inertia can then be constructed on the known conjugate axes, a' and b', according to a method already given [p. 211].

If the given triangle were isosceles, the angles at B and C, would become equal, and the axes, $O\,x$ and $O\,y$, perpendicular to each other, forming the principal axes of the ellipse.

If the triangle were equilateral, the radii of gyration relatively to the three lines joining the vertices with the centre points of the sides would be all equal to each other, and to, $\frac{h}{3\sqrt{2}}$ where, h, is the common perpendicular distance from any vertex to the corresponding base: that is, in this particular instance, the central ellipse would be transformed into a central circle, having a radius,

$$\frac{h}{3\sqrt{2}}$$

Fig. 127.



Examples of Moments of Inertia.

1°. *The Rectangle:*—Required the moment of inertia of a rectangle, whose height is, h, and breadth, b, relatively to the principal axis, $\overline{y\,y}$, Fig. 127.

$$I = \iint x^2\,dx\,dy = \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} x^2\,dx\,dy$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} b\, x^2\, dx = \frac{h^3}{12} \cdot b$$

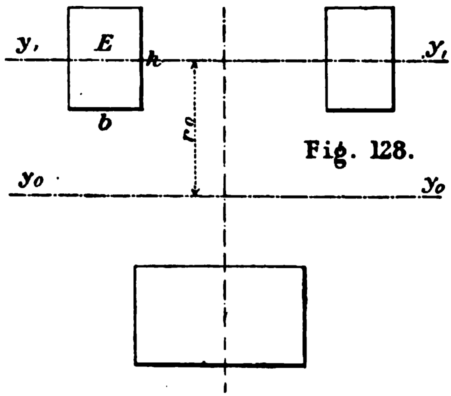
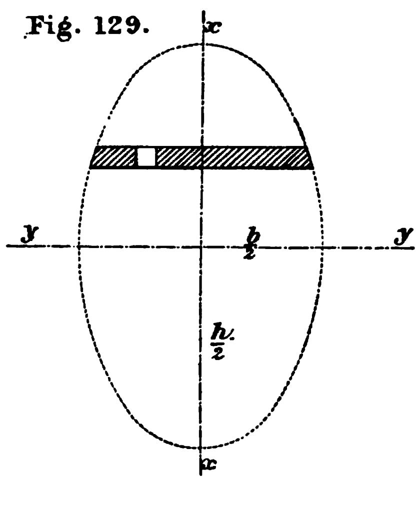


Fig. 128.

2°. *The Ellipse*:—Required the moment of inertia of the ellipse, whose major axis is, h, and minor axis, b, relatively to the principal axis, $y\,y$, Fig. 129.

Fig. 129.



Let

$$\frac{h}{2} = a\,; \; \frac{b}{2} = c\,;$$

then,

$$I = \iint x^2\,dx\,dy = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{y=-\frac{c}{a}\sqrt{a^2-x^2}}^{y=\frac{c}{a}\sqrt{a^2-x^2}} x^2\,dx\,dy$$

$$= \frac{2c}{a}\int_{-\frac{h}{2}}^{\frac{h}{2}} x^2 \sqrt{a^2 - x^2}.\,dx. \quad (1)$$

But,

$$\int x^2 \sqrt{a^2 - x^2}.\,dx = \int x.x\sqrt{a^2 - x^2}.\,dx = \int u.\,dv;$$

where,

$$dv = x\sqrt{a^2 - x^2}.\,dx, \text{ and } v = -\frac{1}{3}[a^2 - x^2]^{\frac{3}{2}}.$$

Hence,

$$\int x.x\sqrt{a^2 - x^2}.\,dx =$$

$$= -\frac{1}{3}x[a^2 - x^2]^{\frac{3}{2}} + \frac{1}{3}\int [a^2 - x^2]^{\frac{3}{2}}\,dx$$

$$= -\frac{1}{3}x[a^2 - x^2]^{\frac{3}{2}} + \frac{1}{3}\int a^2\sqrt{a^2 - x^2}.\,dx$$

$$-\frac{1}{3}\int x^2\sqrt{a^2 - x^2}\,dx.$$

Transposing,

$$\frac{4}{3}\int x^2\sqrt{a^2 - x^2}.\,dx =$$

$$-\frac{1}{3}x[a^2 - x^2]^{\frac{3}{2}} + \frac{1}{3}\int a^2\sqrt{a^2 - x^2}.\,dx$$

When the last equation is taken between the limits, $x = a$, and $x = -a$, the first term of the second member vanishes, and there remains,

$$\int_{-a}^{a} x^2.\sqrt{a^2 - x^2}.\,dx = \frac{1}{4}\int a^2\sqrt{a^2 - x^2}.\,dx$$

$$= \frac{a^2}{4}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\,sin.^{-1}\frac{x}{a}\right],$$

taken within the limits,

$$x = a\,;\; x = -a,$$

giving

$$\int_{-a}^{a} x^2 \sqrt{a^2 - x^2}\, dx = \frac{a^2}{4} \cdot \pi \cdot \frac{a^2}{2}$$

or, since,

$$a = \frac{h}{2},\; = \frac{\pi . h^4}{2 . 64},$$

wherefore, by equation (1).

$$I = \frac{2c}{a} \cdot \frac{\pi . h^4}{2 . 64} = \frac{\pi . b . h^3}{64}$$

The radius of gyration of the same figure, relativbly to the axis, $\overline{yy}$, will be

$$\sqrt{\frac{I}{\pi a c}} = \sqrt{\frac{\frac{1}{64}\,\pi\, b . h^3}{\pi . \frac{h}{2} \cdot \frac{b}{2}}} = \sqrt{\frac{1}{16}\, h^2} = \frac{h}{4} = \frac{a}{2}\,;$$

so that the central ellipse of inertia, having its axes equal respectively to, $\frac{a}{2}$ and $\frac{c}{2}$, is similar and similarly placed to the given ellipse.

3°. *Assemblage of Figures* :—Let there be given a symmetrical assemblage of rectangles, such as is shewn in Fig. 128, and let the dimensions of any given one of these rectangles, E, be, h and b ;—h, being its length parallel to the axis of, x, and, b, its breadth parallel to the axis of, y, the axis, $y_o\,y_o$, being supposed to traverse the general centre of gravity of the system.

The moment of inertia of, E, with respect to an axis, $\overline{y_1\,y_1}$, traversing its own centre of gravity, and parallel to the axis, $\overline{y_o\,y_o}$, will be expressed by [p. 223]

$$\frac{b . h^3}{12}.$$

Let, r_o, be the perpendicular distance between the axes, $\overline{y_1 y_1}$ and $\overline{y_o y_o}$; then the moment of inertia of, E, referred to the general axis, $\overline{y_o y_o}$, will be equal to the sum,

$$I = \frac{b\,h^3}{12} + r_o^2 .\, b\,h$$

Summing the reduced moments of all the rectangular figures comprised in the system, we obtain

$$\Sigma . I = \Sigma . \left[\frac{b\,h^3}{12} + r_o^2 .\, b\,h\right],$$

in which expression, h, b, and r_o, are general symbols, applying to any one of the rectangles, and the sign, Σ, means that we have to add together the various reduced moments, which are all of one form.

In order to make use of the above method, it is necessary to predetermine the centre of gravity of the system of rectangles. For this purpose choose any line as axis of, x, and let, y, be the perpendicular distance of the centre of gravity of any element from the axis chosen. The ordinate y_o, of the general centre of gravity will then be given by the relation,

$$y_o = \frac{\Sigma . E . y}{\Sigma . E},$$

where, E and y, are general symbols, applicable to any figure of the given system,

In like manner, if a second axis of co-ordinates, $\overline{x\,x}$, be taken at right angles to the axis of y,

$$x_o = \frac{\Sigma . E . x}{\Sigma . E}.$$

4°. *Hollow Figures:*—It is sometimes required to find the moment of inertia of a hollow figure, such as that given in Fig. 130 with respect to an axis, $\overline{y_o y_o}$, traversing its centre of gravity. In that case, let, h_1, b_1, be the linear dimensions of the outer, and, h_2, b_2, the corresponding dimensions of the

inner figure. The moment of inertia required will be equal to the difference of the moments of inertia of the outer and inner figures; namely,

$$I = \frac{h_1^3 . b_1}{12} - \frac{h_2^3 . b_2}{12}$$

Fig. 130.

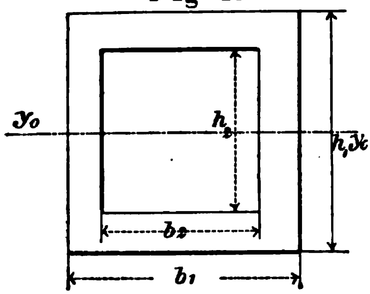
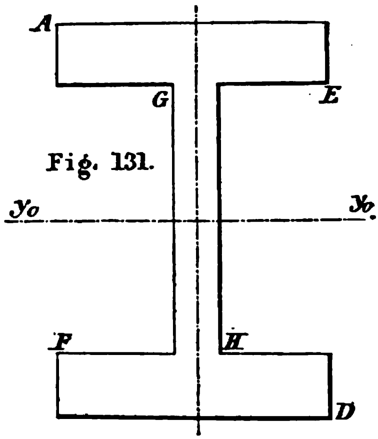


Fig. 131.



5°. *Double, T, Section:*—The moment of inertia of the double, *T*, section, relatively to the principal axis, $\overline{y_o y_o}$, Fig. 131, can be found by successive additions and subtractions as follows,

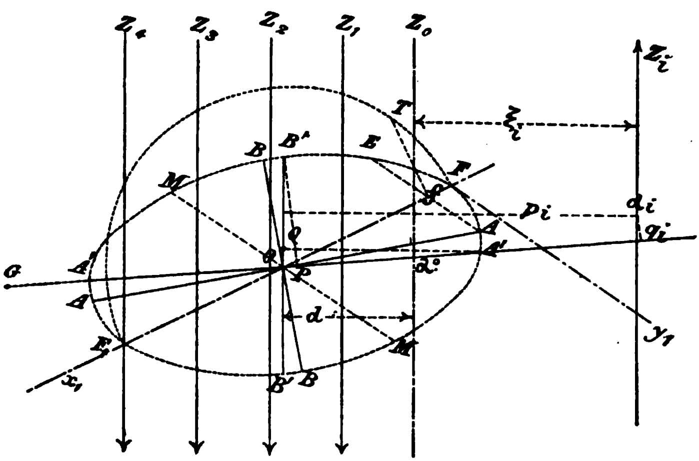
Moment, I, = Moment of, $A D$, rectangle —
— Moment, $E F$, rectangle +
+ Moment, $G H$, rectangle.

In order that this rule should be applicable, it is necessary that the axis, $\overline{y_o y_o}$, traverse the centre of gravity of each of the component rectangles. In cases where the figure becomes unsymmetrical with respect to the axis, $\overline{y_o y_o}$, the moments of the component parts must first be found, relatively to an axis traversing the centre of gravity of each part, and then reduced to the general axis by a previous rule. In all cases where this process is necessary, the centre of gravity of the whole figure must be predetermined. A general axis traversing this centre can then be assumed, parallel to which must be drawn

the independent axes, traversing the centres of gravity of the several component parts.

Central Nuclei of Forces or Stress.—Let there be given a number of parallel forces, applied along the paths, $\overline{Z_1 Z_1}$, $\overline{Z_2 Z_2}$, . . . $Z_i Z_i$ Fig. 132; and let the intensity of these forces vary

Fig. 132.



in proportion to their distances from a neutral line of force, $\overline{Z_0 Z_0}$, contained in the plane of forces. This is an example of uniformly varying stress. Let the perpendicular distance, ξ_i, of the point of application, a_i, measured from the initial line, $\overline{Z_0 Z_0}$, be considered positive to the left and negative to the right of that line, indicating thereby that the forces lying to the right of, $\overline{Z_0 Z_0}$, are opposite in direction to those lying on the left. In connexion with this distinction it is necessary to bear in mind that, since the expression of the moment of inertia involves the *square* of the abscissa of the point of application of any force, this moment takes always the sign of the force.

Further, suppose the letters,

$$k_1, k_2, k_3, \dots k_i,$$

to represent a number of special constants, such that, F_i, being the force applied at, a_i,

$$F_i = k_i \, \xi_i$$

The constants, k_i, may or may not be equal to each other. In the case where they are unequal, each distance, ξ_i, has attached to it a special constant, determined by some known law. In the example, worked out in Fig. 132, the constants, k_i, have been made subject to the same law as the forces, F_i, themselves; being supposed to vary directly as the distances, ξ_i. Hence, in ultimate ratio, the forces, F_i vary as, $k_i \, \xi_i$; that is in the present instance, as ξ_i^2.

Let it be required to find the centre of stress of the system of forces, F_i.

Consider the constants, k_i, as so many forces, having the same points of application as the forces, F_i.

Find by the theory of moments, or by the ordinary graphic process, the centre, O, of the system of forces, k_i.

Trace out the centre-ellipse of inertia, $A\,B\,A'\,B'$, appertaining to the same system, either by the method of line-envelopes [p. 210]; or by finding the moments of inertia, I_1, I_2, I_3, relatively to any three lines, traversing the centre of stress, O, and along the directions of these three lines setting off the respective distances,

$$\delta_1 = \frac{\mu}{r_1}; \quad \delta_2 = \frac{\mu}{r_2}; \quad \delta_3 = \frac{\mu}{r_3},$$

where, μ, is any scalar unit, and

$$r_1 = \sqrt{\frac{I_1}{\Sigma . k_i}}; \quad r_2 = \sqrt{\frac{I_2}{\Sigma . k_i}}; \quad r_3 = \sqrt{\frac{I_3}{\Sigma . k_i}}.$$

The extremities of these line-distances will determine three points on the required ellipse, having for co-ordinates, $x_1\,y_1$, $x_2\,y_2$, and $x_3\,y_3$, relatively to any system of rectangular co-ordinates, assumed as a means of fixing their position.

Now the equation of an ellipse, referred to its centre as

origin and any assumed system of rectangular co-ordinates, can be represented by

$$A y^2 + B x y + C x^2 + 1 = 0,$$

where, $A, B, C,$ are three unknown constants.

By substituting in this equation the co-ordinates of the three fixed points, $x_1 y_1$, $x_2 y_2$, $x_3 y_3$, we shall obtain three linear equations; that is, of the first degree, by aid of which can be determined the values of the three unknown constants, A, B, C. The general equation then assumes a definite form, and can be used as a means to construct the rest of the required curve.

Or, again, having drawn in a part of the ellipse by either of the given methods, draw, in the part constructed, any chord, $\overline{EA}$, bisect it at, f, and through the centre, O, and the point of bisection produce a line, $\overline{FOF_1}$, making

$$F_1 O = F O$$

The line, $\overline{FF_1}$, so found, will be a diameter of the required ellipse.

Next, through, O, parallel to the chord, $\overline{EA}$, draw a line, $\overline{MM}$, to coincide in direction with the conjugate diameter of, $\overline{FF_1}$. It remains only to find the length of the semi-conjugate axis, $\overline{OM}$.

The equation of the ellipse, referred to the axes, $F x_1$, $F y_1$, where, $F y_1$, has been made parallel to, $\overline{MM}$, can be expressed in the form,

$$y_1^2 = \frac{b_1^2}{a_1^2}[2\, a_1 x_1 - x_1^2];$$

hence,

$$b_1 = \overline{OM} = \frac{a_1 y_1}{\sqrt{x_1 (2\, a_1 - x_1)}}$$

$$= \frac{OF . Ef}{\sqrt{Ff\, (2 . OF - Ff)}}$$

This quantity admits of simple graphic expression. Upon

the diameter, FF_1, describe a circle, and from, f, erect a perpendicular intersecting this circle in a point, T;—then

$$fT = \sqrt{Ff.\,(2.\,OF - Ff)},$$

and, therefore,

$$b_1 = \frac{\overline{OF}.\,\overline{Ef}}{fT} = \overline{OM}$$

Hence, $\overline{OM}$, is a fourth proportional, and can be graphically determined by means of similar triangles.

Having determined the conjugate diameters, $\overline{FF_1}$ and $\overline{MM}$, the rest of the figure can be found by the method given at p. 211.

Supposing the ellipse described, draw the diameter, $B'B'$, parallel to the direction of forces, and its conjugate, $A'A'$. Let fall the perpendicular, $B'P$, upon axis, $A'A'$; and, $A'Q$, upon, $B'B'$. Represent by, p_i and q_i, the distances of the point of application, a_i, from axes, $B'B'$ and $A'A'$. respectively. It then follows by a former theorem [p. 210] that,

$$\overline{B'P}^2 = \frac{\Sigma.\,k_i\,q^2_i}{\Sigma k_i}$$

and,

$$\overline{A'Q}^2 = \frac{\Sigma.\,k_i\,p^2_i}{\Sigma.\,k_i}$$

Now, O, being the centre of forces, k_i, the moments of these forces about an axis traversing that centre must, in the sum, be equal to zero; hence

$$\Sigma.\,k_i\,q_i = 0\,;\ \Sigma.\,k_i\,p_i = 0$$

Again, Since, $A'A'$ and $B'B'$, are conjugate diameters, we have the relation,

$$\Sigma.\,k_i\,p_i\,q_i = 0$$

Let p_o, q_o, represent the perpendicular distances of the centre of the forces, F_i, from the axes, $B'B'$ and $A'A'$. Taking moments about, O, we obtain

$$q_o\,\Sigma.\,F_i = \Sigma.\,F_i\,q_i\,;\ p_o\,\Sigma.\,F_i = \Sigma.\,F_i\,p_i.$$

but,

$$\Sigma. F_i q_i = \Sigma. k_i \xi_i q_i = \Sigma. k_i [p - d] q_i = \Sigma. k_i p_i q_i - d. \Sigma. k_i q_i,$$

in which equation, d, is equal to the distance between the axes, $B' B'$ and $\overline{Z_0 Z_0}$.

Both the terms, $\Sigma. k_i p_i q_i$ and $\Sigma. k_i q_i$, vanish, for reasons above given; from which it is to be inferred that,

$$q_o = o,$$

or the centre of the forces, F_i, is situate somewhere on the axis, $A' A'$.

Moreover,

$$\Sigma. F_i p_i = \Sigma. k_i \xi_i p_i = \Sigma. k_i [p_i - d] p_i$$
$$= \Sigma. k_i p^2_i - d \Sigma k_i p_i.$$

but, $\Sigma k_i p_i = O$; hence

$$\Sigma. F_i p_i = \Sigma. k_i p^2_i$$

Further,

$$\Sigma. F_i = \Sigma. k_i \xi_i = \Sigma. k_i [p_i - d]$$
$$= \Sigma. k_i p_i - d. \Sigma. k_i$$
$$= - d \Sigma. k_i,$$

whence,

$$p_o = \frac{\Sigma. F_i p_i}{\Sigma. F_i} = - \frac{\Sigma. k_i p^2_i}{d. \Sigma. k_i} = - \frac{\overline{A' Q}^2}{d}$$

Since, therefore,

$$p_o \times d = - \overline{A' Q}^2,$$

p_o and d, must be of opposite sign, which means that the line, $\overline{Z_0 Z_0}$, and the centre of action of the forces, F_i, are situate on different sides of the axis, $B' B'$.

Let, G, be the required centre of forces, and let angle, $A'\hat{O}B' = \theta$;—then

$$p_o = \overline{OG} \sin. \theta; \text{ and, } A' Q = 0 A' \sin. \theta,$$

whilst, if, a_0, be the point of intersection of the axes, $A' A'$ and $\overline{Z_0 Z_0}$,

$$d = 0a_0 \sin. \theta;$$

Consequently,

$$p_0 \times d = \overline{OG}.\,sin.\,\theta.\,Oa_0\,sin.\,\theta = -A'Q^2 = -\,OA'^2\,sin.^2\theta,$$

or,

$$\overline{OG}.\,Oa_0 = -\,OA'^2\,;$$

the *minus* sign indicating that the points, G and a_0, lie on opposite sides of the centre, O.

The distance, $\overline{OG}$, can be graphically constructed by first finding the graphic equivalent of, OA'^2, according to the following principle and method.

Choose two axes, XX, YY, Fig. 133, at right angles to each other, and set off

$$Ox_0 = \text{unit of scale}$$
$$Ox_1 = OA',$$

Fig. 133.

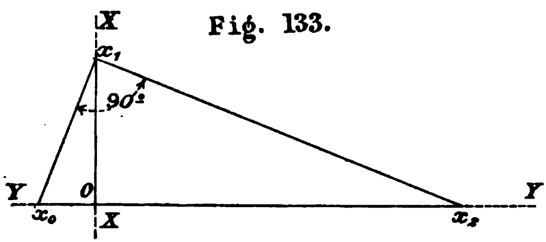
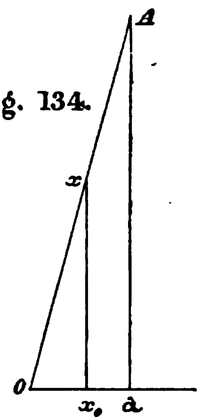


Fig. 134.



then draw, $x_1 x_2$, at right angles to, $x_0 x_1$, meeting the axis of YY, in, x_2. It follows from this construction that

$$Ox_1^2 = OA'^2 = Ox_0.\,Ox_2 = Ox_2$$

Next, Fig. 134, set off

$$Ox_0 = \text{unit of scale}$$
$$Oa = Oa_0 \text{ [Fig. 132]}$$
$$aA = OA'^2 = Ox_2 \text{ [Fig. 133]};$$

then we have, if the lines, $\overline{A\,O}$ and $\overline{x_0\,x}$, be drawn as shewn,

$$\frac{x_0\,x}{a\,A} = \frac{O\,x_0}{O\,a}$$

therefore,

$$\overline{x_0\,x} = \frac{a\,A}{O\,a} = \frac{O\,A'^2}{O\,a_0} = -\,O\,G$$

The point, G, Fig. 132, can then be found by setting off along the axis, $A'\,A'$, produced, and in a direction opposite to, Oa_0, a length,

$$\overline{O\,G} = \overline{x_0\,x}$$

The forces or constants, k_i, will depend on the special nature of the system considered. Take the particular case of a materialised plane acted upon by the forces of gravity alone. The constants, k_i, may in that case express the special densities of the surface at the various points of application, a_i, and they may be either known functions of the co-ordinates of these points, as in heterogeneous bodies; or again constant and equal, as in homogeneous bodies. In both cases the centre of action of the forces, k_i, will coincide with the centre of gravity of the plane surface. Hence is deduced the following general statement:—

If at the various points of a materialised surface there act a series of parallel forces of intensities proportional to the densities of their points of application and the distances of their paths of application from a line, $\overline{Z_0\,Z_0}$, in the same plane; the centre of these forces will be the *antipolar*, G, of the line, $\overline{Z_0\,Z_0}$, determined by the construction already given, and depending on the relation,

$$\overline{O\,G} \times \overline{O\,a_0} = -\,O\,A'^2,$$

where, $O\,A'$, is a certain definite and conjugate semi-diameter of the ellipse of inertia, found in relation to a series of forces, k_i, acting at the various points of the surface, and supposed to be proportional to the densities of the surface at those points. Now, it is manifest that when all the forces lie on the same side of the line of no stress, $\overline{Z_0\,Z_0}$, the forces of

gravity, or other kinds of like forces, will be of one sign. If, however, the line of no stress traverse an area of unlike forces, leaving forces of one sign on the right and others of opposite sign on the left, the forces will change sign in the vicinity of the line, $\overline{Z_0 Z_0}$, and will be of opposite sign on different sides of it.

In the particular case of gravity, the expression for the radius of gyration becomes,

$$r = \sqrt{\frac{I}{\Sigma . k_i}} = \sqrt{\frac{\iint k_i . x^2 . dx\, dy}{\iint k_i\, dx\, dy}}$$

where x, is the abscissa or perpendicular distance of any point in the surface from a neutral axis passing through the centre of stress, O, of the system, k_i, and all the forces are of one sign.

It follows from the form of the expression,

$$\overline{O G} . Oa_0 = - \overline{O A'}^2,$$

in which, $O A'$, is a constant term, that the point, G, will move away from the centre, O, as, a_0, approaches nearer to it; and for any given area, to which the line, $\overline{Z_0 Z_0}$, is tangent, the point, G, will occupy a definite mean position, which may be called, G_1. If the line, $\overline{Z_0 Z_0}$, and with it the point, a_0, move farther away from the centre than this distance, corresponding to the tangency of, $\overline{Z_0 Z_0}$; the point, G, will move nearer to the centre, O, in exactly the same proportion. On the other hand, if the line, $\overline{Z_0 Z_0}$, move nearer to the centre, O, the point, G, will move farther away. If, therefore the line, $\overline{Z_0 Z_0}$, preserving the position of tangency to the given area, be displaced from one point to another of its contour, the point, G, will sympathise with this motion of, $\overline{Z_0 Z_0}$, and occupy successive positions on a curve, which may be called the *antipolar* of the boundary of the given surface. This antipolar curve enjoys the property that, for all centres, G, situate upon it, the zero-lines of stress, $\overline{Z_0 Z_0}$, are tangent to the outer boundary line of the given area, and, for all centres situate within the same curve, the zero-lines lie outside of the plane

surface. In the latter case, all forces, the paths of which are comprised within the given area, are of one sign. On the contrary, for all positions of, G, outside of the antipolar area, the zero-lines of stress are situate within the given materialised surface, subject to stress; and consequently forces applied on one side of this line will be opposite in sign to those applied on the other. When, therefore, in the cross-section of a beam, compression exists on one side of the neutral fibre and tension on the other, the centre of stress lies outside of the antipolar curve, or locus of the points, G, found by determining the antipolar points of lines of stress, tangent to the contour of the area of cross-section.

Fig. 135.

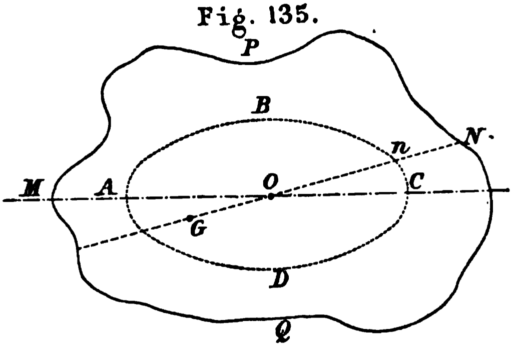
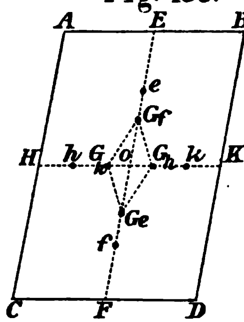


Fig. 136.



The general method, by which to find the limiting antipolar area of any figure, is to construct the central ellipse of inertia relatively to the centre of stress of the surface considered. Suppose that surface to be, $MNPQ$, Fig. 135. Construct the central ellipse of inertia, $ABCD$, and subsequently draw a series of radial lines, $\overline{ON}$, from, O, to the boundary-line of the figure, intersecting the central ellipse in a series of points, n. The point, G, antipolar of, N, will be determined by the relation

$$OG = -\frac{\overline{On}^2}{\overline{ON}}.$$

Take, as an example, the parallelogram, $ABCD$, Fig. 136, of which to find the antipolar curve.

Draw the lines, $\overline{EF}$, and $\overline{HK}$, bisecting respectively the sides, $\overline{AB}$, $\overline{CD}$, and, $\overline{AC}$, $\overline{BD}$. These two lines meet at, O, the centre of stress of the given surface.

The ellipse of inertia, $efhk$, can then be drawn by a previous rule. Next, find the antipolar, G_e, of the line, $\overline{AB}$, by means of the relation,

$$OE.\,\overline{OG_e} = -\,\overline{Oe}^2$$

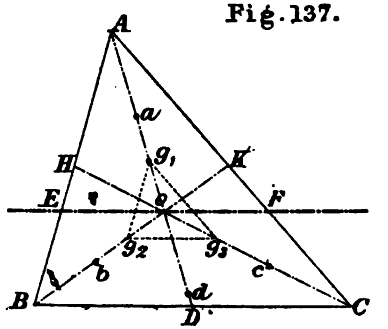
Similarly, determine the point, G_k, antipolar of, $\overline{BD}$, by the equation,

$$\overline{OK}.\,\overline{OG_k} = -\,\overline{Ok}^2$$

In like manner find the antipolars, G_f and G_h, which are placed diametrically opposite to, G_e and G_k, and at equal distances respectively from the centre, O.

The quadrilateral, $G_f\,G_h G_k G_e$, formed by joining the four antipolars named, will furnish the required antipolar curve.

Fig. 137.



Given the triangle, ABC, Fig. 137, find the corresponding antipolar curve.

Draw the line, $\overline{AD}$, from the vertex, A, to the point of bisection, D, of the base, $\overline{BC}$. The centre of gravity of the figure will lie on this line, at a point, O, situate at $\frac{1}{3}$ the perpendicular distance of the vertex, A, from the line, BC.

Through, O, draw a line, $\overline{EF}$, parallel to the base, $\overline{BC}$. Intercepts on the lines, $\overline{OA}$ and $\overline{OF}$, will form conjugate diameters of the ellipse of inertia. Let $O\,a_1$, $O\,b_1$, represent the

semi-conjugate diameters. The antipolar, g_1, of the line, $\overline{B\,C}$, relatively to the point, D, will be determined by the relation,

$$\overline{O\,g_1} = -\frac{O\,d^2}{O\,D}$$

Find in a similar way the antipolars, g_2, g_3, of the lines, $A\,C$ and $A\,B$, respectively. The triangle, formed by joining the antipolars. g_1, g_2, and g_3, will be the central nucleus required.

The antipolar curve of an ellipse is a second ellipse, similar and similarly placed to the first, the radii vectores of the two ellipses being in the ratio of, $\frac{1}{4}$. [Compare, pp. 224–26.]

The central ellipse of inertia of the curved area enclosed between two ellipses, similar and similarly placed, whose axes are respectively, a, b; $m\,a$, $m\,b$, is a third ellipse, similar in every respect to the first two, having for its axes,

$$\frac{a}{2}\sqrt{1+m^2}\,;\quad \frac{b}{2}\sqrt{1+m^2}.$$

And the central nucleus, or antipolar curve, of the same area will be a fourth ellipse, similar to the others with axes equal respectively to

$$\frac{\left(\frac{a}{2}\sqrt{1+m^2}\right)^2}{a}\,;\text{ and }\frac{\left(\frac{b}{2}\sqrt{1+m^2}\right)^2}{b}\,;$$

that is to, $\frac{a}{4}\cdot(1+m^2)$ and $\frac{b}{4}\,(1+m^2)$.

EXAMPLES.

1. If a solid girder, $A\,B$, of 100 feet span, be subjected to a series of nine uniform loads, each equal to 10 tons, concentrated at points, distant respectively, 10, 20, 30, 40, 50, 60, 70, 80, and 90 feet from the abutment, A; and supporting, moreover, the weight of a 40-ton engine, concentrated at a point, 65 feet from, A;—find the bending moment, relatively to a

section, passing vertically through the centre of gravity of the engine.

Bending Moment = 2000 foot-tons.

2. Similarly, find the bending moments, M_x and M_z, in the same system, relatively to sections, distant respectively, 27 ft. and 47 ft., from the left abutment, A.

$M_x = 1350$ foot-tons; $M_z = 1900$ foot-tons.

3. Again, assuming the *ton* as the unit of force, the *foot* as the unit of length, and pitching a *pole* at a point, distant 50 feet from the vertical line of loads; find how many force-units are contained in the intercepts, $\overline{xx}$ and $\overline{zz}$, cut-off by the limits of the polar polygon, on the planes of section, defined in the last example.

$\overline{xx} = 27$; $\overline{zz} = 38$.

4. Under the same conditions, if two polar polygons be drawn, relatively to poles, O_1 and O_2, at 50 ft. and 190 ft., respectively, from the vertical line of loads, shew that the ratio of the intercepts cut-off by the polar polygons, will be

Intercept for pole, O_1, $= 3{\cdot}8$ *Intercept for pole*, O_2.

5. If to the system of forces, brought to bear upon the tree, Fig. 182, Pl. I.. there be added a new force of 300 lbs., due to a gentle wind blowing against the efforts of the men, in a downward direction, at an angle of 15° to the horizon, and intercepting the axis of the tree at a point, 30 ft. above its base;—find the moment about the root, O, due to the wind and combined strength of the men.

Moment = 3720 foot-lbs.

6. Assuming the *pound* as the unit of force, and the *foot* as the unit of length, and pitching a *pole* at a point, distant 8 feet from the resultant on the polygon of forces, find how many

force-units are represented by the intercept, $\overline{x\,y}$, cut-off on a line through, *O*, drawn parallel to the resultant of the system, described in the last example.

$$\overline{x\,y} = 465 \text{ force-units.}$$

7. What is the length of the *arm*, $\overline{O\,P}$, of the moment of the same system, and on which side of, *O*, does it lie?

$$\overline{OP}, \textit{ to windward} = 12{\cdot}92 \text{ feet.}$$

8. If a circular iron rib, $\widehat{C\,D}$, of 100 feet span or chord, and 15 feet versinal rise, be subjected to a series of eight vertical forces, each 10 tons, along paths distant from the left extremity, *C*, by the respective amounts, 10, 20, 30, 40, 50, 60, 70, and 80 feet; and supporting, in addition, a 40-ton engine, whose weight is concentrated at the centre of span, together with the weight of a 20-ton separated tender, concentrated at a point, 70 ft. from, *C*;—find the bending moment at the crown-section of the arch, assuming that the reactions are directed along lines, normal to the circular rib at, *C* and *D*, and that, by means of straight lengthening pieces or *legs*, the abutments are carried down these normal lines, and bedded at points, *A* and *B*, respectively distant from, *C* and *D*, by the equal lengths $C\,A = DB = 17$ feet.

$$\textit{Crown Bending Moment} = 950 \text{ foot-tons.}$$

9. Assuming the *ton*, as the unit of force, the *foot*, as the unit of length, and pitching a pole at, 50 ft., from the line of vertical loads, shew that the intercept, $\overline{x\,y}$, cut-off on a line through the right abutment, *B*, and graphically proportionate to the moment about this point of the resultant of the system of external forces (1—8), applied to this circular iron rib, will be, in the absence of reactions,

$$\overline{x\,y} = 177 \text{ force-units.}$$

10. In the same example, shew that the bending moment, produced at a section, 38 ft. from the left abutment, *A*, by

the vertical action of the load of 50 tons, concentrated at the centre of span, is *half* the moment, produced at the same section, by the whole series of forces and vertical reactions.

11. P and Q, are fixed points on the circumference of a circle, QA and QB, are any chords at right angles to each other, on opposite sides of, QP, shew, by the aid of the polar polygon and polygon of forces, that, if QA and QB represent forces both in magnitude and direction, the moments of the resultants of the pairs of forces, about, P, are equal and constant.

12. Shew by the graphic method, that if the sum of the moments of a series of forces, acting in one plane on a particle retains the same value for two points in the plane, the straight line, joining these two points, must be parallel to the line of action of the resultant force.

13. Let, ABC, be a triangle, D, E, F, the middle points of the sides opposite to A, B, and C, respectively; shew, by aid of the polar and force polygons, that forces, represented in magnitude and direction, by AD, BE, and CF, constitute a system in equilibrium.

14. Find by a direct method the Moment of Inertia, about the central horizontal axis, of a hollow rectangle, whose inside and outside breadths and lengths are, b_1, h_1, and b, h, respectively.

$$\textit{Moment of Inertia} = \frac{1}{12}\,(b.h^3 - b_1 h_1^3)$$

15. Shew that the expression for the Moment of Inertia, about the central horizontal axis, of a I section, the flanges of which are of the external breadth, b, the breadth of the web, $b - b_1$, the depth to outside of flanges, h, and to inside flanges, h_1, is identical in form with the result given in the last example.

16. Given a cross-form of section, whose vertical arm is, h, units in height, and, b, in breadth; and horizontal arm, h_1, and, b_1, units; shew that its Moment of Inertia about the central horizontal axis is

$$I = \frac{1}{12}\,\{b_1 h_1^3 + b\,(h^3 - h_1^3)\}$$

17. Find the Moment of Inertia of a solid circular section relatively to its diameter, $2r$.

$$\textit{Moment} = \tfrac{1}{4}.\,\pi.\,r^4.$$

18. Find the Moment of Inertia, relatively to an axis passing through the centre of gravity, of a simple T section, the flange of which has a breadth, b, and depth, h; and the web a depth, h_1, and breadth, b_1.

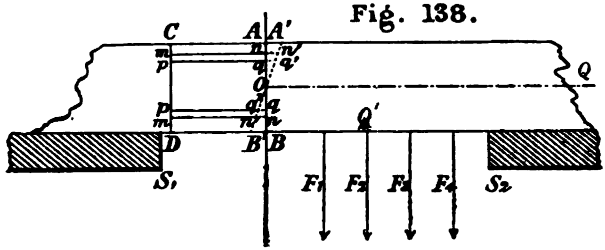
$$I = \tfrac{1}{3}\,\{b\,[x^3 - (x - h)^3] + b_1\,[x + (h_1 - x)^3]\}$$

CHAPTER VI.

STRAIGHT BEAMS AND GIRDERS.

1°. THE DEFLECTION OF BEAMS UNDER STRESS.—Let a plane section, $\overline{A\,B}$, Fig. 138, be made through a straight beam resting on the supports, S_1, S_2. Suppose the plane of the paper so to divide the beam, that the resultant of the applied loads, F_1, F_2, F_3, F_4, may act in a line in that plane. Further, let the plane of the paper be a diametral plane, dividing the cross-section into two equal and symmetrical parts.

Fig. 138.



The forces, F, being brought to bear upon the straight beam, will deflect it vertically; but this deflection will not materially alter the position of the trace of cross-section, $\overline{A\,B}$, relatively to the horizontal central line, $\overline{O\,Q}$, to which before deflection it was normal. After deflection, $\overline{A\,B}$, will be shifted into a position, $A'\,B'$, still normal to the deflected centre-line, $\overline{O\,Q'}$.

When the beam is deflected, each element of the cross-section $\overline{A\,B}$, suffers a certain displacement, parallel to the line, $\overline{n\,q'}$, in the figure, and approximately parallel to, $\overline{O\,Q}$, representing the centre-line of the beam in its position before deflection.

In virtue of this displacement an internal stress is induced in the beam, which expresses the effort of the section to right itself. This effort will act parallel to the line, $\overline{n\,q'}$, and can therefore be resolved into two component parts, one, f, parallel to the line of vertical forces; and a second, t, parallel to the centre-line of the beam, $\overline{O\,Q}$.

Summing the stresses, f, resulting from the various points in the cross-section displaced, we obtain an expression of the form, $\Sigma.f$, which, since the beam is in equilibrium, must be equal to the sum, $\Sigma.F$, of the vertical forces applied between the cross-section, $\overline{A\,B}$, and the right end, S_2, of the beam. [Pt. I. Ch. V. § 1]. Hence,

$$\Sigma.f = \Sigma.F$$

where, $\Sigma.F$, includes the reaction-force at the support, S_2, and $\Sigma.f$, is termed the shearing force at the given cross-section.

Further, it will be seen that for small deflections the angle which $\overline{n\,q'}$, makes with the vertical is approximately, 90°. Consequently the value of the cosine, made use of to resolve the effort along, $\overline{n\,q'}$, in the direction of the stresses, f, will be proportionately small; and consequently the stresses, f, themselves, will be small compared with the stresses, t.

Project the forces, F, parallel to the centre-line, $\overline{O\,Q}$. It will then be seen that

$$\Sigma.F.\cos.90° = o = \Sigma.t.$$

Let us examine the meaning of this equation. Consider two neighbouring sections of the beam, $\overline{A\,B}$, and, $\overline{C\,D}$. It has been shewn how the various points in the cross-section, $\overline{A\,B}$, suffer displacement parallel to the line, $\overline{n\,q'}$. If these displacements be resolved parallel to the centre-line, $\overline{O\,Q}$; their projections will be proportionate to the stresses, t. Now, the relation, $\Sigma.t = O$, expresses the fact that the sum of these projections is equal to zero; that is, the lines, $\overline{m\,n}$, connecting, $\overline{A\,B}$ and $\overline{C\,D}$, are extended by amounts, $\overline{n\,n'}$ above the centre, O, and contracted by equal amounts below the same point, in such a manner that the sum of the extensions and contractions

vanishes. Hence, there must exist in the cross-section a certain point, O, which suffers neither extension nor contraction, and above it lines such as, $\overline{m\,n}$, are lengthened by an amount, $\overline{n\,n'}$ and lines, equally distant below it, are shortened by an equal amount.

Let the vertical distance from, O, of the element, $n\,q\,n'\,q'$, be, y, and let $\overline{m\,n} = L$; angle $AOA' = \theta$. Suppose, moreover, that the elasticity of the section in the vicinity of the element, $\overline{n\,q}$, be represented by E.

Under these conditions the elemental elongation, $\overline{n'\,n}$, will be equal to

$$y.\,\theta,$$

and this having taken place in a length of beam equal to, $\overline{m\,n}$ or L, the elongation per unit of length will be expressed by

$$\frac{y.\,\theta}{L}.$$

The corresponding stress, t, will therefore be expressed by [eq. 3. p. 84]

$$E.\,\omega.\,\frac{y.\,\theta}{L} = t$$

hence,

$$\Sigma.\,t = o = \Sigma.\,E.\,\omega.\,\frac{y\,\theta.}{L}. \qquad (1)$$

The term, $\frac{\theta,}{L}$ is common to all the elements, ω, of the cross-section, and is, moreover, constant.

If the cross-section be of a homogeneous nature, the factor, E, will also be a constant quantity: so that the equation already given assumes the form,

$$\Sigma.\,t = o = E.\,\frac{\theta.}{L}\,\Sigma\,\omega.\,y.$$

If the centre of gravity of the section be distant, y_o, from the centre of zero-stress, O, then

$$y_o = \frac{\Sigma\,\omega.\,y}{\Sigma.\,\omega}.$$

Therefore, since by the preceding equation, $\Sigma. \omega\, y = O$, it follows that, $y_o = o$, and the centres of zero-stress and gravity coincide.

The locus of points, O, or the line-locus of the centres of gravity of consecutive sections of the beam, is called its *neutral axis.*

The axis passing through, O, perpendicularly to the plane of the paper and longitudinal section of the beam, is termed the *axis of flexure* relatively to the cross-section considered.

In order that the beam may be in equilibrium, it is necessary not only that

$$\Sigma. f - \Sigma. F = O, \text{ and } \Sigma. t = O,$$

which are the two conditions just stated; but also that the sum of the moments, taken about some axis perpendicular to the plane of the forces, should be *nil.*

Let, therefore, the axis of flexure passing through, O, be chosen as the axis of moments, and let, x, be the distance from this axis of any of the parallel forces, F, applied between, O, and either end of the beam, the reaction at that end being included. Then, if, y, be the vertical co-ordinate of any local stress, t, induced at the section, we must have

$$\Sigma. F x - \Sigma. t. y = O \qquad (2)$$

the term, $\Sigma. Fx$, is called the bending moment at the section considered, and, $\Sigma. t. y$, the moment of resistance.

Let, $+ T$, represent the sum of all stresses, t, above the centre, O; then, the sum of the stresses below the same point will be, $- T$, and if, y_o, equal the vertical distance between the points of application of these two resultants, T and $-T$, we shall have

$$T. y_o = \Sigma. y. t \qquad (3)$$

Again, let x_o, be the abscissa of the resultant of the forces, F; say, on the right of the plane of section; then

$$x_o. \Sigma. F = \Sigma. Fx \qquad (4)$$

Consequently, by equations [(2), (3) and (4)]

$$T. y_o = x_o. \Sigma. F$$

The form of this equation confirms the truth of an observation already made that, $\Sigma. F$, is generally small compared with T; for by the above relation,

$$\frac{\Sigma. F}{T} = \frac{y_0}{x_0};$$

but, y_0, depends on the depth of the cross-section, and x_0, on the length of the longitudinal section of the beam. As a rule, therefore, $\Sigma. F$, is much less than, T.

Take, $\Sigma. Fx = M$. Equation (2) furnishes the relation

$$M = \Sigma. t. y$$

Now, it has been proved that [p. 246]

$$t = E. \omega\, y. \frac{\theta}{L}$$

hence,

$$M. = \Sigma. \left[E. \omega. y. \frac{\theta}{L}\right]. y,$$

which, if, E, be constant, can be put into the form,

$$M = E. \frac{\theta}{L}. \Sigma. \omega. y^2$$

The term,

$$\Sigma. \omega\, y^2 = \iint y^2\, dx\, dy = I,$$

is the moment of inertia of the section relatively to the axis of flexure. Substituting the symbol, I, for $\Sigma. \omega. y^2$, there results,

$$M = E. \frac{\theta}{L}. I$$

and,

$$\frac{\theta}{L} = \frac{M}{EI}; \qquad (5)$$

that is, the angular deflection per unit-length of the beam equals, $\frac{M}{EI}$.

Further,

$$t = \mathrm{E}.\ \omega.\ y.\ \frac{\theta}{L} = E\ \omega.\ \frac{M}{E\,I}.\ y$$

$$= \frac{M.\ y.\ \omega}{I};$$

wherefore,

$$\frac{t}{\omega} = \frac{M.\ y}{I}$$

The value $\frac{t}{\omega}$, expresses the stress per unit-area of cross-section of the beam. It depends for its absolute value on that of the ordinate, y; and, therefore, the maxima-values of, $\frac{t}{\omega}$, for the same section, correspond to the maxima of, y. These occur at the upper and lower limits of the cross-section. Let O, be situate at the centre of figure of the cross-section, supposed to be rectangular, and let, h, be the depth of the section. The greatest tensions and thrusts will take place in the extreme fibres of cross-section, and will be equal to each other, and to,

$$t = \frac{M}{I} \cdot \frac{h}{2}.$$

This part of the subject is of such vast importance in the calculation of the strength of beams that it will be advisable to develop the demonstrations, so as to bring into clearer light the relations subsisting among the symbols used.

Take the beam shewn in Fig. 139, and, before deflection, suppose the sections, $\overline{A\,B}$ and $\overline{C\,D}$, to be separated by the distance, $\overline{T\,O} = L$.

If the beam be then deflected, under the action of specified loads, through an angle, $B\hat{O}B' = \theta$, the absolute extension of the fibre, $p\,m\,n\,q$, of original length, L, is equal to $y.\,\theta$, where y, represents the ordinate, $O\,q$, approximately.

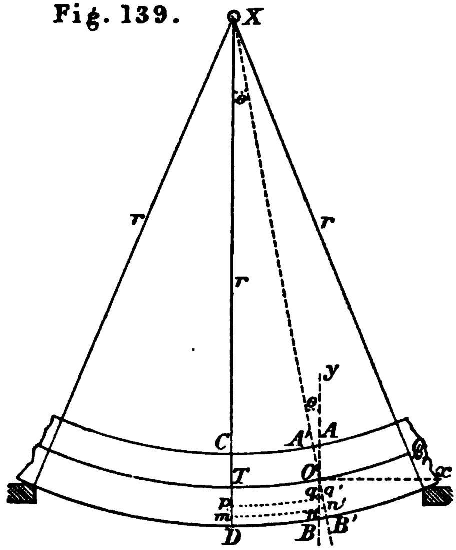
The elongation per unit of length of beam is, therefore,

$$\frac{y.\ \theta}{L}$$

The tension induced in the unit-area, ω, of cross-section at, q, can be expressed by [p. 246].

$$E.\omega.\frac{y\theta}{L} = t$$

Fig. 139.



The moment of this local tension relatively to the axis of flexure through, O, will be

$$E.\ \omega.\frac{y\theta}{L}.\ y$$

Now, in the limit, the section of the fibre, $p\,m\,n\,q$, becomes indefinitely small, and then we shall have,

$$y\theta = \overline{q\,q'}\,;\ L = \overline{p\,q}\,;\ y = \overline{O\,q}.$$

Hence the above expression for the elemental moment, due to the local stress, t, at q, can be put into the form,

$$E\,\omega.\frac{q\,q'}{p\,q}\ .\ O\,q$$

Produce the line, $A'\,B'$, to meet the trace of section, $\overline{C\,D}$, in, X.

By similar triangles

$$\frac{q\,q'}{O\,q} = \frac{O\,T}{T\,X} = \frac{p\,q}{r},$$

in which, $r = \overline{T\,X}$, represents the radius of curvature corresponding to the final deflection of the beam.

Substituting for $q\,q'$, its equivalent, $\frac{p\,q}{r}.\,O\,q$, the value of the elemental moment becomes,

$$E.\omega.\frac{q\,q'}{p\,q} \times O\,q = E.\,\omega.\,\frac{O\,q^2}{r}$$
$$= E\,\omega.\frac{y^2}{r}.$$

It has been already shewn that the sum of the elemental moments due to the longitudinal stresses, t; namely, $\Sigma.\,t\,y$, is equal to the sum of the moments of the forces applied between the plane of section, $\overline{A\,B}$, and one extremity of the beam. This equality has been put in the form, [p. 248],

$$M = \Sigma.\,t\,y,$$

and can be interpreted to mean that the bending moment, M, at any section is equal to the moment of resistance at the same section.

Consequently, summing all the elemental moments of the form, $E\,\omega.\frac{y^2}{r}$, discussed above, we obtain

$$\Sigma.\,E\,\omega.\frac{y^2}{r} = \frac{E}{r}.\,\Sigma.\,\omega.\,y^2 = M.$$

that is,

$$\frac{E\,I}{r} = M.$$

Since [eq. (5), p. 248],

$$E\,I\,.\,\frac{\theta}{L} = M,$$

it may be deduced that

$$\frac{\theta}{L} = \frac{1}{r}$$

or,

$$r.\theta = L,$$

which relation is made self-evident by an inspection of the figure.

Let a horizontal line through, O, be taken as the axis of, x, and a line, perpendicular to the last-mentioned, as the axis of, y.

The radius of curvature of the deflected centre-line will then be expressed according to the usual formula by

$$r = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

and the curvature by

$$\frac{1}{r} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

But, since the curve assumed by the deflected centre-line differs only slightly from a horizontal line, the symbol, $\frac{dy}{dx}$, or tangent of the angle formed with the axis of, x, by a line drawn to touch the curve at any point, will represent a very small quantity, and its square may be therefore neglected, when compared with unity.

Making, $\frac{dy}{dx} = o$, the expression for the curvature becomes,

$$\frac{1}{r} = \frac{d^2y}{dx^2},$$

and the equation of moments,

$$\frac{EI}{r} = M,$$

takes the form,

$$\pm EI . \frac{d^2 y}{d x^2} = M.$$

The double sign is prefixed; because it implicitly exists in the denominator of the fraction expressing the value of, $\frac{1}{r}$. If it be agreed to consider those moments, M, positive, which tend, as in the figure, to turn the axis of, x, towards that of, y; the quantity, $\frac{dy}{dx}$, taken relatively to points in the curve having positive abscissæ, increases with the deflection in a downward sense; and therefore its differential coefficient, $\frac{d^2 y}{d x^2}$, must be affected with a positive sign. Establishing this convention, the equation of moments may be written,

$$E. I. \frac{d^2 y}{dx^2} = M.$$

When, E, is not a constant quantity; that is, when the section is of a heterogeneous nature, the term, $\Sigma . t = O$, bears the following interpretation,

$$\Sigma . E \omega . \frac{y \theta}{L} = o = \frac{\theta}{L} . \Sigma (E \omega) . y.$$

In this case the axis of flexure would traverse the centre, O, coinciding with an ideal centre of gravity, found on the supposition that each element, ω, of cross-section has a density, proportionate to the local coefficient of elasticity. The equation of moments would then become,

$$\frac{\theta}{L} \Sigma (E \omega) . y^2 = M;$$

hence

$$\frac{\theta}{L} = \frac{1}{r} = \frac{M}{\Sigma (E \omega) . y^2}$$

From this equation it is evident that the values of, r and $\Sigma (E \omega) y^2$, increase together, so that the greater the value of,

$\Sigma\,(E\,\omega)\,y^2$, the less is the curvature of the beam. In this sense the quantity, $\Sigma\,(E\,\omega)\,y^2$, is said to measure the difficulty experienced in bending the beam, and for that reason has been termed its ***moment of inflexibility.*** When, E, is constant, the moment of inflexibility takes the simpler form, $E\,I$. When, $\frac{\theta}{L} = 1$; that is, when $r = 1$, the same term expresses the bending moment at the section. Again; this moment of inflexibility, in its general form, may be defined to be the moment of inertia of the cross-section of the beam relatively to the axis of flexure, supposing each elemental mass to be equal to the elemental area, ω, multiplied by the local elasticity, E, of the point considered.

2. Beams of Uniform Strength.—It has been shewn [p. 249], that the unit-stress, $\frac{t}{\omega}$, at any point of a cross-section of a beam, supporting vertical loads, can be expressed by

$$\frac{t}{\omega} = \frac{M.\,y}{I}$$

And; since, for the same cross-section and system of loads, M and I, are two constants, the stress per unit of section varies only with the ordinate, y, measuring the height or depth of the point considered, above or below the axis of flexure. Let, y_m, be the maximum value of, y; then

$$\frac{t_m}{\omega} = \frac{M.\,y_m}{I},$$

is the maximum stress induced in the extreme fibre of the cross-section, most distant from, O, the centre of stress.

The quantity, $\frac{t_m}{\omega}$, has a special value for each cross-section, and it may be required so to arrange the different cross-sections that the various maxima, $\frac{t_m}{\omega}$, may be equal to each other and to some constant value, called the safe working stress of the material out of which the beam is made. In this case the

beam is said to be of *equal resistance*, which means that the greatest stress brought to bear in any cross-section has a constant and equal value. This condition of stress can be created by varying the form of the cross-section according to a law, which will now be explained.

Suppose a beam, Fig. 140, fixed at one end, $\overline{DE}$, and supporting at its free extremity, F, a weight, W. Take any cross-section, $\overline{AB}$, at a distance, x, from the free extremity, F; and let the form of section be a rectangle of a depth, h, and a breadth, b.

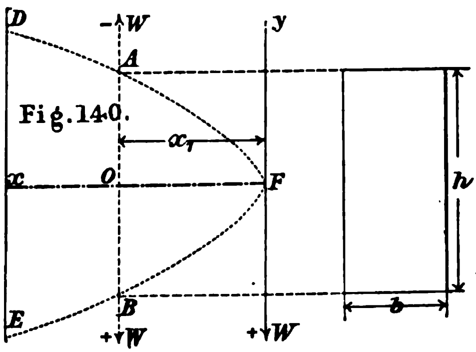


Fig. 140.

Further, suppose two equal and opposite forces, W, to be applied along the line of section, $\overline{AB}$. The addition of these two forces, which balance each other, will not alter the condition of equilibrium in which the beam exists.

The upward force, W, along, $\overline{AB}$, and the downward force, W, at, F, constitute a moment or couple, tending to wrest away the portion, $\overline{AFB}$, and turning it from left to right. This moment will be equal to the common force, W, multiplied by the arm, x; that is,

$$M = W.x$$

The remaining downward force, W, acting along, $\overline{AB}$, expresses the amount of shearing force brought to bear upon the section. The moment of inertia of the rectangular cross-section relatively to the axis of flexure is,

$$\frac{b.h^3}{12}$$

The greatest value of the ordinate, y, will be, $\frac{h}{2}$, and consequently the expression,

$$\frac{t_m}{\omega} = \frac{M.y_m}{I}.$$

becomes,

$$\frac{t_m}{\omega} = \frac{W.x.\frac{h}{2}}{\frac{b\,h^3}{12}} = \frac{6.\,W.x}{b\,h^2}$$

Now, if the beam is to be of equal resistance,

$$\frac{t_m}{\omega} = \frac{6\ W.x}{b\,h^2} = a\ constant.$$

Suppose, in this case, that the depth, h, is made to vary so as to satisfy the last equation at all cross-sections of the beam; whilst the breadth, b, remains constant. It follows that,

$$\frac{x}{h^2} = \frac{b \times constant}{6.\ W} = \frac{b.\ R}{6.\ W.},$$

where, R, represents the safe working stress of the material employed. Since, moreover, all the factors in the expression, $\frac{b.\ R}{6\ W}$, are constant for all sections of the beam, we can put

$$\frac{b.\ R}{6\ W} = C;$$

hence,

$$\frac{x}{h^2} = C;\ \text{and}\ h^2 = \frac{1}{C}.x$$

Take,

$$\frac{h}{2} = h_1;$$

then by the last equation

$$\frac{x}{C} = 4\,h_1^2$$

and therefore,

$$h_1^2 = \frac{1.}{4\,C} \cdot x = 4\,a_1\,x,$$

where,

$$\frac{1}{4\,C} = 4\,a_1$$

The equation, $h^2_1 = 4\,a_1\,x$, shews that the curve, assumed by the boundary limits of the varying depths of cross-sections, is a parabola; and since, when $x = o$, $h_1 = o$, the depth, h, vanishes at the point of suspension, F. It would, nevertheless, be found necessary to strengthen the end of the beam so as to resist the action of shearing forces, W, which is the same at all sections.

If the load, instead of being suspended at the end, F, were uniformly distributed over the length of the beam; p, representing the load per unit of length, the elemental moment induced at the given section by any unit-load, p, applied at a distance, x, from the origin, F, would be equal to

$$p.\,x\,d\,x,$$

and the sum of all such unit-moments, acting between, O, and, F, will be

$$\int_o^{x_1} p\,x\,dx = \frac{p.\,x_1^2}{2},$$

where,

$$x_1 = \overline{OF}.$$

In this particular case,

$$\frac{t^m}{\omega} = \frac{M\,y_m}{I} = \frac{\frac{p.\,x_1^2}{2} \cdot \frac{h}{2}}{\frac{b\,h^3}{12}} = \frac{3\,p\,x_1^2}{b\,h^2} = R\,;$$

or, if, b, remain constant

$$\frac{x_1^2}{h^2} = \frac{b.\,R}{3.\,p} = C'^2.$$

Hence,

$$\pm\frac{x_1}{h} = C.$$

Let,

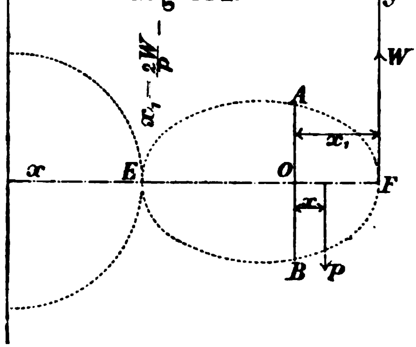
$$h = 2\,h_1\,;$$

then,

$$h_1 = \pm\frac{x_1}{2\,C}.$$

The form of the last equation shews that the limits of, h_1, above and below the centre-line of the beam, are two straight lines, meeting at the outer extremity, F. Wherefore, the longitudinal section of the beam will be a triangle.

Fig. 141.



Let the beam, Fig. 141, support an uniform load, p, per unit of length, as well as a concentrated load, W, applied at, F, in an upward sense, opposed to that of the forces, p.

The bending moment, M, at any section will be that due to the difference of the moments arising from the concentrated and distributed loads separately taken. Therefore, considering upward forces positive and downward forces negative,

$$M = W x_1 - \frac{1}{2}p.\,x_1^2$$

The constant maximum stress will then be

$$\frac{t}{\omega} = \frac{M.y_m}{I} = \frac{\left[W x_1 - \frac{1}{2} p.x_1^2\right] . \frac{h}{2}}{\frac{b\,h^3}{12}}$$

$$= \frac{3\,[2.\,W.\,x_1 - p\,x_1^2]}{b\,h^2} = R.$$

Hence, if, b, remain constant,

$$2\,W x_1 - p.x_1^2 = \frac{R.b}{3}.h^2 = C.\,h^2$$

Let, $h = 2\,h_1$, then by substitution and transposition,

$$4\,C.\,h_1^2 + p.\,x_1^2 - 2\,W.\,x_1 = o,$$

which, since $B^2 - 4\,A\,C = o - 16\,C\,p$, is the equation to an ellipse, referred to the extremity of its major axis as origin.

The same equation can be put into the form,

$$x_1\,[2\,W - p\,x_1] = 4\,C\,h_1^2,$$

which shews that, $h_1 = o$, for $x_1 = o$, and for, $x_1 = \frac{2\,W}{p}$; that is, the ellipse-curve meets the axis of, x, at those points.

From, $M = W\,x_1 - \frac{1}{2} . p\,x_1^2$, we obtain

$$M = x_1\left[W - \frac{1}{2}\,p\,x_1\right];$$

wherefore, it will be seen that, M, begins to be negative beyond the point for which, $x_1 = \frac{2\,W}{p}$

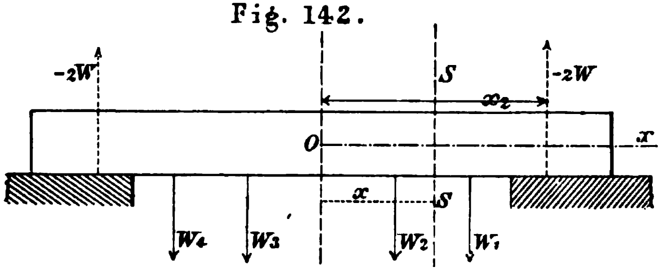
The outline of the longitudinal section of the beam will be that shewn in the figure. The major axis of the ellipse, as already demonstrated, is equal to, $\overline{FE} = \frac{2\,W}{p}$. After the point, E, has been reached, owing to the factor, $[2\,W - p\,x_1]$, remaining negative, the equation, assumes the form,

$$4\,C\,h_1^2 - p\,x_1^2 + 2\,W\,x_1 = o,$$

which is the equation to a hyperbola, referred to the point, E, as origin, which is, therefore, the vertex common to the two curves, at which the bending moment is *nil*. A shearing force, nevertheless, exists at the section through this vertex, and consequently sufficient area must be allowed to resist the action of this force.

3. BENDING MOMENTS AND SHEARING FORCES.—Let co-ordinates be measured from any point, O, on the centre line of a beam, [Fig. 142]. Let, moreover, W_1, be any one of a system of forces applied between the section, $\overline{SS}$, and the right extremity of the beam.

Fig. 142.



The abscissa of W_1, being x_1, and that of the sectional plane, x; the bending moment induced at $\overline{SS}$, will be equal to

$$M = \pm \Sigma . [W_1 (x_1 - x)];$$

where, Σ, represents a summation extending from the sectional plane considered to the right end of the beam. This sum includes the moments of all forces between the section, $\overline{SS}$, and the extremity, as well as the moment of the reaction at that point. Consequently, the bending moment will take the sign of the resultant of the included system of forces, and the sign of this resultant will depend upon the positions of the loads and the plane of section. For this reason the double sign, $\pm$, has been prefixed to the bending moment.

The shearing force at the same section will be given by the sum,

$$F = \pm \Sigma . W_1;$$

where, W_1, is a general symbol including all the forces and reactions acting between, $\overline{SS}$ and the end of the beam.

But, differentiating the given expression for the bending moment,

$$\frac{d\,M}{d\,x} = \mp \Sigma.\, W_1 = -F.$$

Hence, it may be inferred that, making abstraction of the sign, the shearing force at any section is equal in absolute value to the differential coefficient of the bending moment; or

$$F = \frac{d\,M}{d\,x}.$$

The sign of, F, may be of the same or opposite kind to that of, $\frac{d\,M}{d\,x}$. It will be of opposite kind when a force, W, considered positive in the sum, $\Sigma.\, W_1$, gives rise to a moment in the sum, $\Sigma\,[W_1\,(x_1 - x)\,]$, which is also made positive. To take a simple example, let the downward forces, W_1, W_2, W_3, and W_4, Fig. 142, be positive; the upward forces or reactions, $2\,W$, acting at the ends of the beam, be negative;—the moments due to positive forces, tending to turn the body in the same sense as the hands of a watch revolve, be considered positive; whilst moments of an opposite tendency are made negative.

Taking the sum of the moments between the sectional plane, $\overline{SS}$, and the end of the beam, we obtain,

$$\begin{aligned} M &= W_1\,(x_1 - x) - 2\,W\,(x_2 - x) \\ &= W_1\,x_1 - W_1.\,x - 2\,W\,x_2 + 2\,W\,x. \end{aligned}$$

Let all the forces, W_1, W_2, W_3, W_4, be equal to each other and to, W; then

$$M = W.\,x - W.\,(2\,x_2 - x_1)\,;$$

wherefore,

$$\frac{d\,M}{d\,x} = W.$$

But,

$$F = \Sigma.\ W_1 = -\ 2\ W + W_1 = -\ W;$$

consequently,

$$F = \frac{-d\,M}{d\,x}.$$

It will be seen, therefore, that the signs of, F, and $\frac{d\,M}{d\,x}$, are of opposite kind, for the reason that it has been agreed that positive forces, such as W_1, should give rise to positive moments in the sum, $\Sigma.\ [W_1(x_1 - x)]$; and that negative upward forces, such as $-\ 2\ W$, should create negative moments.

The above investigation depends on the supposition that the bending moment comes exclusively from the effect of certain definite, applied loads; and, therefore, the relation, $F = \frac{d\,M}{d\,x}$, no longer holds, if we suppose the bending moment to be due in part to the sum of a succession of small elemental moments of the kind, $\mu.d\,x$; in such manner that the general expression of the bending moment will contain the term

$$\int_x^{x_2} \mu.\,dx.$$

The addition of a number of small moments, μ, due to pairs of equal and opposite infinitesimal forces distributed over the length of the beam, would not affect the value of the shearing forces, which would still be equal to the sum of the independent vertical forces.

On the other hand, if we differentiate the bending moment, now composed of the terms,

$$M = \pm\ \Sigma.\,[W_1(x_1 - x)] \pm \int_x^{x_2} \mu\,dx,$$

the value of $\frac{d\,M}{d\,x}$, so found, will be either greater or less than that of, F, which depends only on the first term of the right

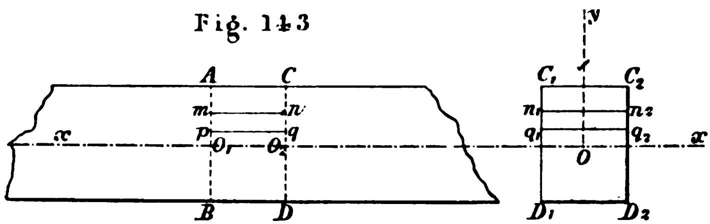
hand member of the equation. Integrating the second term, there results,

$$\int_{x}^{x_2} \mu\, dx = \mu\,(x_2 - x),$$

the differential coefficient of which is equal to, $-\mu$; hence, to find the shearing force, a quantity, $\pm\,\psi\,\mu$, must be added to the value of, $\frac{d\,M}{d\,x}$, as deduced from the given equation.

4. DISTRIBUTION OF SHEARING FORCES.—Let, $\overline{A\,B}$, $\overline{C\,D}$, Fig. 143, be traces of two adjoining cross-sections of a beam. The side view of section, $\overline{C\,D}$, is shewn, at, $C_1\,C_2\,D_1\,D_2$. Let, y, be the vertical distance of the line, $q_1\,q_2$, from the centre-line

Fig. 143



of the beam taken as the axis of x. It has previously been shewn that the stress per unit of surface at this part of the cross-section is given by the expression

$$\frac{t}{\omega} = \frac{M.\,y}{I}.$$

Represent the constant breadth of cross-section by

$$q_1\,q_2 = b\,;$$

so that the area of the elemental surface, $n_1\,n_2\,q_1\,q_2$, will be equal to,

$$b \times d\,y,$$

and the stress applied over this area,

$$\frac{M}{I}\,.\,y\,.\,b\,dy.$$

The sum, therefore, of the stresses, exerted between the limits, y, and the ordinate, y_1, of the extreme fibre, $\overline{C_1 C_2}$, can be put into the integral form,

$$T = \frac{M}{I} \cdot b \int_{y}^{y_1} y \, dy.$$

Now, it is evident that longitudinal stress of the kind, T, must create a tendency in the fibres to slide, one over the other. This tendency will be least at the upper and lower boundaries, $C_1 C_2$, $D_1 D_2$, and will go on increasing towards the neutral axis traversing, O, where the tendency in the fibres to slide relatively to each other is greatest. That such is the case is manifest; for the stress, T, determined above, is the resultant of the longitudinal stresses between, $\overline{q_1 q_2}$ and $\overline{C_1 C_2}$, and therefore its point of application will be found somewhere within the same limits. Its immediate effect will be an effort to shift the part of beam, $A\,p\,q\,C$, along the plane, $p\,q$. This effort is in proportion to the intensity of the stress, T, and is greatest when the integral expression for, T, is taken between the limits, y_1 and O;—that is, when the effort made results in a tendency to move the part, $A\,O_1\,O_2\,C$, along the axis of, x. In order to find the absolute value of the stress to which this tendency to longitudinal movement is due, it is necessary to remember that, M, varies from one cross-section to the other, along the axis of, x.,—whereas, owing to the uniformity existing in the transverse dimensions of the sections, the terms, b and I, are constant. The value of, T, will, therefore, depend on the variable, M, alone; and the increase or decrease in, T, in the interval, $d\,x$, separating two adjoining sections, $\overline{A\,B}$ and $\overline{C\,D}$, will be a measure of the resultant stress acting between the sections, as well as of the tendency in the fibres to slip one over the other. This increase or decrease in the value of, T, in passing from one section to another, along the axis of x, will be represented by, $d\,T$; where,

$$\frac{d\,T}{d\,x} = \frac{d\,T}{d\,M} \cdot \frac{d\,M}{d\,x}.$$

But,

$$\frac{dM}{dx} = F, \text{ and } \frac{dT}{dM} = \frac{d}{dM} \cdot \left[\frac{M}{I} \cdot b \int_y^{y_1} y\, dy\right].$$

Wherefore,

$$\frac{dT}{dx} = \frac{F}{I} \cdot b \int_y^{y_1} y\, dy\,;$$

which is the absolute value of the stress or effort made to move the part of beam, $A\,p\,q\,C$, tangentially to the plane of, $p\,q$. If no movement actually results, it is a sign that the cohesion among the fibres is at least sufficient to resist the effort to slide. When the cohesion, C, is sufficient, and no more than sufficient, to overcome the sliding effort, there exists the equality,

$$C = \frac{dT}{dx} = \frac{F}{I} \cdot b \int_y^{y_1} y\, dy,$$

which is an expression for the cohesion *per unit of length of the beam.* This, divided by the breadth of section b, will give the cohesion *per unit of length and breadth* of the beam ; or

$$c = \frac{C}{b} = \frac{F}{I} \cdot \int_y^{y_1} y\, dy \qquad (6)$$

The existence of this tendency to longitudinal slip has long been a matter of practical observation, and accounts for the fact that, when a beam is made up of several planks, these are rigidly connected together either by means of through bolts, or assisted by stepped joints, in opposing the effort made to shift them lengthwise relatively to each other.

Consider an elemental prism of the beam, $m\,n\,p\,q$, contained between two planes, $\overline{A\,B}$ and $\overline{C\,D}$, perpendicular to the axis of x ; and two planes, $\overline{m\,n}$ and $\overline{p\,q}$, parallel to the same line. Let, as before,

$$\overline{p\,q} = dx\,;\; \overline{n_1 q_1} = dy,$$

and take moments, about an axis traversing the point, m, perpendicularly to the plane of the paper, of all the forces applied

to the elemental prism, $m\,n\,p\,q$. These forces are;—1°. Its weight, which, if $b = q_1\,q_2$, and, ω, represent the weight of the material per unit of volume, may be expressed in the form,

$$b.\ \omega.\ dx\,dy.$$

The point of application of this force will coincide with the centre of gravity of the prism, which is at a distance, $\frac{d\,x}{2}$, from the point, m;—hence, the moment due to weight will be,

$$\frac{b.\,\omega}{2}.\ dx^2.\,dy,$$

or an infinitesmal quantity of the third order. 2°. There will be certain normal forces developed against the horizontal surfaces, through, $\overline{n_1\,n_2}$, and $\overline{q_1\,q_2}$. These normal forces exist by reason of cohesion, in the same way as it can be concluded from the presence of friction between two surfaces that there must exist corresponding normal pressures. The total normal force acting againt one of the faces, through $\overline{n_1\,n_2}$ or $\overline{q_1\,q_2}$, will be in proportion to the area of the face, equal $b\,d\,x$, and the local force of cohesion, c; so that it may be put into the form,

$$\phi\,[b.\,c.\,dx.]$$

Since the material employed in the beam is of homogeneous texture, the cohesion may be supposed constant throughout the length of the elemental prism, and the point of application of the normal force will then act at a distance, $\frac{d\,x}{2}$, from the point, m; the moment due to this force will therefore be

$$\phi\,[b.\,c.\,dx].\,\frac{d\,x}{2}.$$

Now, the moments of normal forces due to cohesion acting against the surfaces, $n_1\,n_2$, $q_1\,q_2$, are similar in kind; but, since the local cohesion, c, induced, varies from one point to another in a vertical direction, [see equation (6) p. 265], these two

moments will differ slightly in amount, and their difference, seeing that they act in opposite senses, will be equal to the resultant moment, due to normal forces, tending to produce rotation in the prism about the axis through, m.

The expression for this resultant moment can be obtained from the moment,

$$\phi\ [b.\,c.\,dx]\,\frac{dx}{2},$$

by taking the differential,

$$\frac{d}{dy}.\ \phi.\ [bc.\ dx].\frac{dx}{2} = \frac{dx}{2}.\ \phi'\ [bc\ dx].\ \frac{dc}{dy};$$

but by equation, 6,

$$\frac{dc}{dy} = \frac{F}{I}\,y\,dy;$$

hence the quantity,

$$d.\ \phi\ [bc.\ dx].\frac{dx}{2},$$

contains terms of at least the third infinitesimal order. 3°. There are normal stresses acting against the sections, $\overline{mp}$ and $\overline{nq}$. The stress at either of these sections will be proportionate to the special value of, t, at that point of the beam, and to the area of the elemental surface considered, viz., $b.\ dy$. In other terms the stress will be equal to the product,

$$b.\ t.\ dy.$$

The *arm* of this stress relatively to the axis of moments through, m, is, $\frac{dy}{2}$; so that the corresponding moment will be,

$$\frac{b}{2}.\ t.\ dy^2,$$

and its differential, with respect to t,

$$\frac{b}{2}.\ dt.\ dy^2,$$

which is an expression for the resultant moment of the

difference of stresses, t, acting against the faces, $\overline{mp}$ and $\overline{nq}$. It will be seen to contain infinitesimals of the third order. 4°. There are tangential stresses applied along the four faces of the prism, $\overline{mn}$, $\overline{mp}$, $\overline{pq}$, and $\overline{nq}$. The stresses, acting in the planes, $\overline{mn}$ and $\overline{mp}$, pass through the axis of moments, and their moments vanish. To obtain the moments of the stresses lying in the planes, $\overline{pq}$ and $\overline{nq}$, let, f, represent the shearing stress per unit of the surface, $n_1\, n_2\, q_1\, q_2$; and, c, the longitudinal resistance to sliding per unit of length and breadth of the surface of, pq, as before determined.

The total tangential stress in the plane, nq, will be equal to,

$$bf.\, dy\,;$$

similarly, the total stress in the plane, $\overline{pq}$, will be equal to,

$$bc.\, dx.$$

The respective moments of these two stresses relatively to the axis traversing the point, m, will be,

$$bf.\, dy.\, dx\,; \text{ and, } bc.\, dx.\, dy.$$

Neglecting orders of infinitesimals higher than the second, and equating to zero the sum of the moments of all the forces acting on the elemental prism, which exists in a state of equilibrium, we find,

$$bf.\, dy\, dx = bc\, dx\, dy\,;$$

or,

$$f = c$$

But the value of the unit-cohesion, c, has been previously determined; therefore, f, the shearing force per unit-area of cross section, is also known; that is,

$$f = c = \frac{F}{I} \int_{y}^{y_1} y\, dy \qquad \text{(See eq. 6.)}$$

As an example, take a beam of rectangular cross-section;—height equal to, h, and breadth to, b.

Then,

$$y_1 = \frac{h}{2}; \; I = \frac{b\,h^3}{12};$$

consequently,

$$f = \frac{F}{2I}\left[\frac{h^2}{4} - y^2\right]$$
$$= \frac{6.F}{b\,h^3}\left[\frac{h^2}{4} - y^2\right];$$

and, when $y = o$; that is at the centre of the section,

$$f = \frac{3}{2} \cdot \frac{F}{b.h};$$

When, $y = \frac{h}{2}$; that is at the upper and lower limits of the cross-section,

$$f = o.$$

Again, the *mean* unit of shearing stress would be equal to,

$$f_o = \frac{\textit{Total Shearing Force}}{\textit{Area of Cross-Section}} = \frac{F}{bh}$$

The *maximum*, $f_m = \frac{3}{2} \cdot \frac{F}{b\,h}$, exceeds the *mean* unit of shearing stress, and they bear to each other the proportion expressed by

$$\frac{f_m}{f_o} = \frac{\frac{3}{2} \cdot \frac{F}{b\,h}}{\frac{F}{b\,h}} = \frac{3}{2}$$

Let there be given a beam of double *T* section, as shewn in Fig. 144, let the depth of this section be, *h*, and the depth, *a*, of each flange be small in comparison with, *h*. Moreover, let it be granted that the sum of the areas of the two flanges constitute the greater part of the total area of the section, the area of the web being comparatively small.

Under these conditions the area of cross-section may be con-

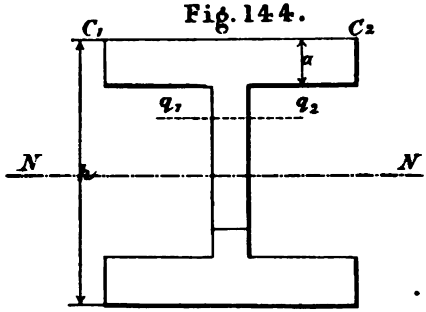
sidered to be concentrated at two points, on opposite sides of the neutral axis, $\overline{NN}$, and at equal distances from it, $\frac{h}{2}$.

In this case,

$$I = 2\left[\tfrac{1}{2}A.\left(\frac{h}{2}\right)^2\right] = \frac{A}{4}.h^2$$

where, A, represents the sum of the areas of the flanges. To find the total stress, T, between any line, $\overline{q_1, q_2}$, traversing the web, and the upper extremity, $\overline{C_1 C_2}$, of the flange, it is neces-

Fig. 144.



sary to recall to mind that the stress, t, at any point in the section, distant, y, from the neutral axis, $\overline{NN}$, is equal to,

$$\frac{M}{I}.y;$$

which, for $y = \frac{h}{2}$, bears the maximum value,

$$\frac{M}{I}.\frac{h}{2}.$$

This maximum unit stress may be taken as the stress existing at any point in the section, which supposition involves a slight error made on the safe side. Remembering that the area of the web is small compared to the sum of the areas of the flanges, A, we find the total stress applied between, $\overline{NN}$ and $\overline{C_1 C_2}$, to be equal to

$$\text{maximum unit-stress} \times \frac{A}{2} = \frac{M}{I}.\frac{h}{2} \times \frac{A}{2}.$$

But, it has been shewn that

$$\frac{A}{I} = \frac{4}{h^2};$$

hence,

$$T. = \frac{M}{I} \cdot \frac{A.h}{4} = \frac{M}{h}.$$

Thus the sum of the longitudinal stresses between the neutral axis and the top or bottom of the beam is equal to the bending moment at the section divided by the total depth of the beam.

Differentiating this resultant stress, which varies with, M, we obtain

$$\frac{d\,T}{d\,M} = \frac{1}{h},$$

and multiplying both sides of this equation by, $\frac{d\,M}{d\,x} = F$, there results

$$\frac{d\,T}{d\,x} = \frac{F}{h},$$

which is an expression for the increase or decrease of longitudinal stress between two adjoining sections, separated by the interval, dx.

The area of a surface whose breadth is the breadth of the web, β, and length, dx, is equal to the product, $\beta.dx$. Dividing therefore the total increment or decrement of longitudinal stress, $dT = \frac{F}{h}.dx$, by the area, $\beta\,dx$, we obtain the unit of resistance, c, distributed over the horizontal and longitudinal section, to which has been given the name *cohesion.* Hence,

$$\frac{d\,T}{\beta.\,dx} = c = f = \frac{F}{h}.\,dx \div \beta.\,dx$$

$$= \frac{F}{\beta\,h},$$

which equation shews that the shearing force, F, may be looked upon as distributed over the area of the web only ;—for it has

been allowed that the depth, a, of the flanges, is small in comparison with, h.

In the example of a beam of rectangular cross-section already mentioned, [p. 268], the dimensions, h and b, were supposed invariable. The depth, h, may, however, be taken to vary for different sections of the beam; whilst the maximum stress, $\frac{M.h}{2\,I}$, remains constant. This is the case in solids of equal resistance. In this case the total stress applied between the neutral axis and the upper or lower limit of the beam will be strictly,

$$T = \frac{M}{I}\int_0^{\frac{h}{2}} b\,y\,d\,y = \frac{3}{2}\cdot\frac{M}{h}.$$

Here the stress is expressed as a function of two variables M and h, connected by the relation,

$$\frac{M}{I}\cdot\frac{h}{2} = R,$$

where, R, is the safe working stress per unit-area of cross-section ; wherefore,

$$\frac{6.\,M}{b.\,h^2} = R,$$

and,

$$M = \frac{b\,R}{6}\,.\,h^2 = C.\,h^2.$$

Differentiating the equation of stress,

$$T = \frac{3}{2}\frac{M}{h},$$

we obtain,

$$d\,T = \frac{3}{2}\left[\frac{d\,M}{h} - \frac{M.\,dh}{h^2}\right];$$

but by the above relation between, M and h,

$$log.\,M = 2\,log.\,h + log.\,C,$$

wherefore,

$$\frac{d\,M}{M} = \frac{2\,dh}{h}$$

and,

$$\frac{M.\,dh}{h} = \frac{d\,M}{2};$$

so that,

$$d\,T = \frac{3}{2}\left[\frac{d\,M}{h} - \frac{d\,M}{2\,h}\right]$$

$$= \frac{3}{4}.\frac{d\,M}{h} = \frac{3}{4}.\frac{F.\,dx}{h};$$

the stress per unit of surface is, therefore,

$$\frac{d\,T}{b.\,dx} = c = f = \frac{3}{4}.\frac{F}{b\,h};$$

or half that found on the supposition that the section was constant throughout. Solids of equal resistance enjoy, therefore, advantages in respect to the distribution of shearing stress.

5. THEORY OF THE SUPERPOSITION OF FORCES.—When a girder is placed in position, the loads applied to it give rise to reactions at the two end supports, as well as at the intermediate ones, if such exist. For example, the weight, P, concentrated at the point, L, on a beam, supported at, S_1 and S_2, gives rise to reactions at the supports, which can be determined by the relation,

$$P : R_1 : R_2 : : \overline{S_1 S_2} : \overline{L S_2} : \overline{L S_1} \qquad \text{[Fig. 145.]}$$

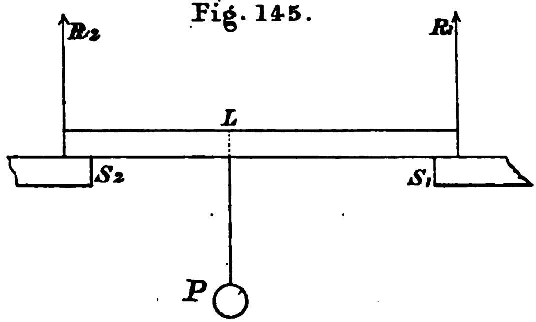
When, however, the beam is fixed at one end, and free and unsupported at the other, as shewn in Fig. 146, a change takes place in the relative senses of the reactions, R_1 and R_2, one of these being directed upwards and the other downwards.

According to the principle of the lever, the proportions above stated still obtain. This being so, the line, $\overline{L S_2}$, can be taken to graphically represent the greater reaction at, S_1; the

line $\overline{L S_1}$ to represent the reaction, R_2; whilst, $\overline{S_1 S_2}$, will typify the amount of the load at, L. Hence,

$$R_1 = \overline{L S_2} = [\overline{L S_1} + \overline{S_1 S_2}] = R_2 + \overline{S_1 S_2}$$

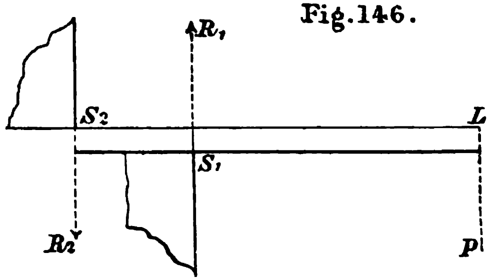
Fig. 145.



Considering the reaction, R_1, as thus made up of the sum $[R_2 + \overline{S_1 S_2}]$; it is clear that the first term of this sum, namely: R_2, acting at, S_1, coupled with the reaction, R_2, acting at, S_2, will constitute a force-couple, whose moment is equal to

$$R_2 . \overline{S_1 S_2} .$$

Fig. 146.



The second term of the expression, $[R_2 + \overline{S_1 S_2}]$; namely, $\overline{S_1 S_2}$, combined with, $P = \overline{S_1 S_2}$, acting at L, will give a second kind of moment applied at, S_1, equal in amount to the load, P, multiplied by the arm, $\overline{L S_1}$. Consequently at all points of support of the kind, S_1, which are called *encastrements*, there exists not only a reaction of the ordinary kind; but as well a differential couple, tending to preserve the line, $\overline{S_1 S_2}$, rigidly straight and horizontal.

Let a girder be continously supported at any number of points, S_1, S_2, S_3, ... S_n, Fig. 147, horizontally distant from the origin, O, by the amounts. b_1, b_2, b_3, ... b_n, the reactions at these points being represented by, R_1, R_2, R_3 . . . . . R_n, Let the abscissæ of the paths of application of a number of loads, P, applied to the girder, be represented by, ξ_1, ξ_2, ξ_3, ... ξ_n. Let, moreover, the radius of gyration of any cross-section of the beam, distant, x, from the origin, be expressed as usual by

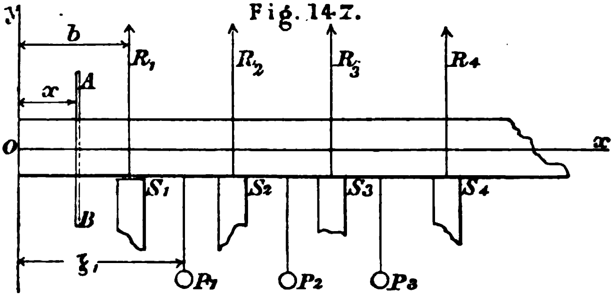
$$r^2. \Sigma. E\,\omega = \Sigma. E\,\omega. y^2$$

that is,

$$r^2 = \frac{\Sigma. E. \omega. y^2}{\Sigma. E\,\omega};$$

where, ω, represents the area of an element, $dy\,dx$, of the cross-section ;—E, the local elasticity of the same element ;—

Fig. 147.



and, y, its distance above or below the centre-line of the beam ; or more correctly from the axis of flexure.

If under these conditions, M, represent the moment brought to bear upon a section, $\overline{A\,B}$, distant, x, from the origin, O, and all the supports be supposed at first *simple* ; that is, not *encastrements*,

$$M = P_1 [\xi_1 - x] + P_2 [\xi_2 - x] + \ldots + P_n [\xi_n - x]$$
$$- R_1 [b_1 - x] - R_2 [b_2 - x] + \ldots - R_n [\xi_n - x]$$

If, however, some of the supports, S_1, S_2 . . . S_n, are *encastrements*, there must be added to the above value of, M, a series of moments, representing the moments induced at the

encastrement-supports. Let, $\mu_1, \mu_2, \mu_3, \ldots \mu_n$, typify the moments. The total effect of the couples, μ, upon the beam, looked upon as a whole, will be equal to their algebraic sum; so that the expression for, M, takes the more general form,

$$\begin{aligned} M = P_1 [\xi_1 - x] + P_2 [\xi_2 - x] + \ldots + P_n [\xi_n - x], \\ - R_1 [b_1 - x] - R_2 [b_2 - x] - \ldots\ldots - R_n [b_n - x] \\ + [\mu_1 + \mu_2 + \mu_3 + \ldots\ldots + \mu_n], \end{aligned}$$

where any or all of the terms, $\mu_1, \mu_2, \mu_3 \ldots \mu_n$, may be positive, negative, or zero quantities. It should be remarked that in this investigation an assumption has been implicitly made that the moments, μ, which are supposed to act in the same plane, can be shifted into any position in their plane of action, without altering the general effort they exercise to rotate the beam. This assumption is perfectly legitimate when the body acted upon is free to move in space; the question here is whether it be equally licit when the body is influenced by a combination of moments arising, not from forces freely acting upon a body at liberty to turn in space, but from a series of loads limited in their effect by reactions induced at a number of fixed points. It seems, therefore, necessary to make a very important distinction. If the moments, μ, are produced by, or owe their existence to the series of loads, P, and the consequent reactions, R, the above assumption is perfectly legitimate; since it only amounts to the usual condition that the sum of the moments, directly due to the forces, P and R, and of the additional moments indirectly due to the same forces, by reason of the *quasi-encastrements* created at the supports, should, when the beam is in equilibrium, vanish. But, if the beam or girder be intentionally fixed down upon any of the supports, S, so as to create a *rigid encastrement*, sufficient in itself to resist the action of the loads applied on both sides of it, it would seem that, the continuity of the beam being absolutely broken, the above assumption no longer represents the actual condition of stress, and each section of the girder, so divided, would have to be considered alone and separate from the others, treating each span as a beam fixed

at both ends, or only at one end, accordingly as both ends, or only one end, were fastened down.

Returning to the general equation, and allowing that it represents the condition of stress in a continuous girder, it will be observed that in obtaining the value of, M, the sum of the moments of the loads, reactions, and independent moments has been taken between the limits corresponding to the section, $\overline{A\,B}$, and the right end of the beam. Now, while the origin remains fixed at, O, suppose the plane of section, $\overline{A\,B}$, to be movable, and let it travel in the direction of the right-hand extremity of the girder.

Following the course of the plane in this movement, we perceive that it will pass in turn beyond the lines of action of the forces and reactions, P and R, leaving them one after another on its left. For example, when the plane of section, moving forward, has reached a point distant, ξ_1, from the origin, the term, $P_1\,[\xi_1 - x]$, will disappear; for P_1, being afterwards situate to the left of the plane of section, cannot again appear in the expression for, M.

Now, let

$$\chi_1 = [\xi_1 - x]\,;\ \beta_1 = [b_1 - x],$$

and similarly,

$$\chi_n = [\xi_n - x]\,;\ \beta_n = [b_n - x],$$

then the general equation giving the value of, M, takes the form

$$\begin{aligned} M = \chi_1 . P_1 + \chi_2 P_2 + \ldots\ldots + \chi_n P_n, \\ -\beta_1 R_1 - \beta_2 R_2 - \ldots - \beta_n R_n \\ + \mu_1 + \mu_2 + \mu_3 + \ldots\ldots + \mu_n, \end{aligned}$$

in which expression any symbol, χ_i, is subject to the condition that for values of, x, less than, ξ_i,

$$\chi_i = [\xi_i - x],$$

whilst for values of, x, greater than, ξ_i,

$$\chi_i = o.$$

A similar condition attaches to the symbol, β_i, since when

$$x < b_i,$$
$$\beta_i = [b_i - x],$$

and when,

$$x > b_i,$$
$$\beta_i = o.$$

Again any moment, μ_i, is to be admitted for all values of, x, up to $x = b_i$, the abscissa of the encastrement at which, μ_i, is supposed to act, and suppressed for all values of, $x > b_i$.

By reason of the bending moments, M, acting at various cross-sections of the beam, its neutral fibre will undergo deflection. Each of the cross-sections, of which the beam is made up, will in turn be identified with the plane of section, as it travels from the left to the right extremity of the beam. The general equation to the curve formed by the deflected neutral fibre has already been derived and expressed in the form, [pp. 252—53],

$$\Sigma.\ E\,\omega.\ y^2.\ \frac{d^2 y}{d\,x^2} = M.$$

Let, $\Sigma.\ E\,\omega = e$; then,

$$e\,\gamma^2 = \Sigma.\ E\,\omega.\ y^2,$$

and

$$e\,\gamma^2.\ \frac{d^2 y}{dx^2} = M\,;$$

that is,

$$\frac{d^2 y}{dx^2} = \frac{1}{e\,\gamma^2}[\chi_1 P_1 + \chi_2 P_2 + \ldots + \chi_n P_n]$$
$$- \frac{1}{e\,\gamma^2}[\beta_1 R_1 + \beta_2 R_2 + \ldots + \beta_n R_n]$$
$$+ \frac{1}{e\,\gamma^2}[\mu_1 + \mu_2 \ldots + \mu_n].$$

By a first integration,

$$\frac{dy}{dx} = P_1\int_o^x \frac{\chi_1}{e\,\gamma^2}\,dx + P_2\int_o^x \frac{\chi_2}{e\,\gamma^2}.\,dx + \ldots P_n\int_o^x \frac{\chi_n}{e\,\gamma^2}.\,dx$$

$$-R_1\int_o^x \frac{\beta_1}{e\,r^2}.\,dx - R_2\int_o^x \frac{\beta_2}{e\,r^2}dx - \ldots - R_n\int_o^x \frac{\beta_n}{e\,r^2}\,dx$$

$$+\mu_1\int_o^x \frac{1}{e\,r^2}\,dx + \mu_2\int_o^x \frac{1}{e\,r^2}\,dx + \ldots + \mu_n\int_o^x \frac{1}{e\,r^2}\,dx + C,$$

in which equation, C, represents the value of $\frac{dy}{dx}$, when $x = o$, corresponding to the initial position of the movable plane, at, O. It may be symbolised by, $C = \frac{dy_o}{dx_o}$.

Integrating a second time we obtain,

$$y = P_1\int_o^x\int_o^x \frac{\chi_1}{e\,r^2}.\,dx.\,dx. + P_2\int_o^x\int_o^x \frac{\chi_1}{e\,r^2}.\,dx\,dx + \ldots$$

$$+ P_n\int_o^x\int_o^x \frac{\chi_n}{e\,r^2}.\,dx.\,dx - \Bigg[R_1\int_o^x\int_o^x \frac{\beta_1}{e\,r^2}.\,dx.\,dx$$

$$+ R_2\int_o^x\int_o^x \frac{\beta_2}{e\,r^2}.\,dx\,dx + \ldots + R_n\int_o^x\int_o^x \frac{\beta_n}{e\,r^2}.\,dx\,dx\Bigg] +$$

$$+\mu_1\int_o^x\int_o^x \frac{1}{e\,r^2}\,dx\,dx + \mu_2\int_o^x\int_o^x \frac{1}{e\,r^2}.\,dx\,dx + \ldots$$

$$+\mu_n\int_o^x\int_o^x \frac{1}{e\,r^2}.\,dx\,dx + \frac{dy_o}{dx^o}.\,x + C_1,$$

in which expression, $C_1 = y_o$, represents the value of, y, when, $x = o$.

In the above equations, for $\frac{d\,y}{d\,x}$ and y, we find a number, n,

of terms, of the type, R, expressing the unknown reactions at the supports; and a number, say k, of terms of the kind, μ, representing the unknown moments induced at the supports constituting encastrements. It has, previously, been shewn that each encastrement involves not only an unknown reaction, R, but as well an unknown attendant couple, μ. Consequently, there are in all, $\overline{n + k}$, unknown quantities connected with the, n, points of support, besides the two unknown constants, y_o and $\frac{d\,y_o}{d\,x_o}$. The sum of these undetermined values is, therefore, $\overline{n + k + 2}$. It becomes necessary, in order to make the general equation of any use, to determine these unknown quantities by means of relations existing between them and the known forces. Since the system is supposed to be in equilibrium, the algebraic sum of the vertical forces must be zero; hence

$$P_1 + P_2 + \dots + P_n - [R_1 + R_2 + \dots + R_n] = O$$

This equation expresses the absence of all vertical movement; it remains to state the condition that there is no movement of rotation about any axis at right angles to the plane of the forces. Taking moments about an axis traversing the origin, O, we find

$$\begin{aligned} P_1\,\xi_1 + P_2\,\xi_2 + \dots + P_n\,\xi_n \\ - [R_1 b_1 + R_2 b_2 + \dots + R_n b_n] \\ + \mu_1 + \mu_2 + \mu_3 + \dots + \mu_n = O \end{aligned}$$

The moments, μ, are supposed to be shifted, according to an assumption already explained, and to be applied in the vicinity of the origin, O, one force of each couple passing through the axis of moments.

Owing to the rigidity of the supporting pillars, S, there is an absence of all deflection of the neutral fibre at those points. If, therefore, the original horizontal line of the neutral fibre, before deflection, had been taken as the axis of, x, the values of, y, corresponding to the abscissæ, b_1, b_2, b_3, . . . b_n, must vanish.

Employing the equation for, y, [p. 279] and stating its nullity for the above, n, values of, x, there will result, n, additional equations, which, added to the two equations of equilibrium already derived, make in all, $n + 2$, equations. The remaining, k, equations are deduced from the general expression for, $\frac{dy}{dx}$; for, it is manifest that $\frac{dy}{dx}$, must vanish at each of the, k, points of support, where there is an encastrement.

Having determined in this way the, $n + k + 2$, unknown quantities, and having introduced their corresponding values into the general equations for, M, $\frac{dy}{dx}$, and, y, those equations will then furnish the bending moment, M, the tangent of inclination, $\frac{dy}{dx}$, and the ordinate of deflection, y, of the neutral fibre at any cross-section, distant, x, from the origin, O.

When the cross-sections of the beam are uniform and of constant dimensions, the term, $e\,r^2$, is also constant, and may be placed outside the sign of integration. This simplifies the integrations of the general forms; for example, take the term,

$$P_i \int_o^x \int_o^x \frac{\chi_i}{e\,r^2}\, dx . dx = \frac{P_i}{e\,r^2} \int_o^x \int_o^x \chi_i . \, dx\, dx.$$

By definition,

$$\chi_i = \xi_i - x \,; \text{ so long as } x \leqq \xi_i,$$

and

$$\chi_i = o \,; \text{ when } x \geqq \xi_i.$$

Hence, so long as $x \leqq \xi_i$,

$$\int_o^x \chi_i\, dx = \int_o^x (\xi_i - x)\, dx = \xi_i\, x - \frac{x^2}{2} \cdot$$

But, if the plane of cross-section be situate at a part of the beam, for which, $x > \xi_i$, let, χ_i, be represented by an ordinate, χ, as shewn in Fig. 148; this ordinate, up to a certain point, determined by,

$$x = \xi_i,$$

will be expressed as a certain function of, x, namely,

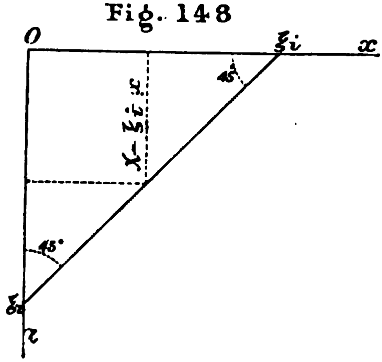
$$\chi = \xi_i - x.$$

After the point, $x = \xi_i$, has been reached, χ_i, is constantly zero. The integration of

$$\int_0^x \chi_i . dx,$$

must, therefore, be taken in two parts, since, χ_i, is not the

Fig. 148



same kind of function of, x, *continuously* from, $x = x$, to, $x = o$; but changes its nature at the point corresponding to

$$x = \xi_i.$$

It will be seen that since the equation,

$$\chi = (\xi_i - x) = -x + \xi_i,$$

represents a straight line inclined at, 45°, below the axis of, x, and with equal intercepts, ξ_i, on the axes of, x and χ, the functional form, χ_i, can be graphically represented by the line, $\xi_i\,\xi_i$, Fig. 148.

Summing the two parts of the integration, we obtain,

$$\int_0^x \chi_i\, dx = \int_0^{\xi_i} [\xi_i - x]\, dx + \int_{\xi_i}^x o.\, dx = \frac{\xi_i^2}{2}.$$

Taking now the double form of integration, the case is simple, if the abscissa, x, of the cross-section considered be such that

$$x \leqq \xi_i;$$

for, then, we shall have,

$$\int_o^x \int_o^x \chi_i \, dx \, dx = \int_o^x [\xi_i x - \frac{x^2}{2}] \, dx = \frac{\xi_i x^2}{2} - \frac{x^3}{6}.$$

But, if the abscissa of cross-section be such that

$$x > \xi_i;$$

then, again, the integration must be taken in two parts; for, if we make

$$\int_o^x \chi_i \, dx = V,$$

it follows, by what has just been shewn, that, V, has not the same functional form, in terms of, x, for all values of that coordinate, comprised between the limits of the plane of cross-section and the origin at, O. Up to the limit corresponding to

$$x = \xi_i,$$

V, has the functional form, $\xi_i x - \frac{x^2}{2}$.

Beyond this point, V, is no longer a function of, x, at all; but is constant and equal to, $\frac{\xi_i^2}{2}$. The integration of the expression, $V\,dx$, must, therefore, be taken in two parts; so that,

$$\int_o^x V.\,dx = \int_o^{\xi_i} [\xi_i . x - \frac{x^2}{2}].\,dx + \int_{\xi_i}^x \frac{\xi_i^2}{2}.\,dx$$

$$= \frac{\xi_i^2}{2}[x - \frac{\xi_i}{3}].$$

The curve, corresponding to the values of, V, in terms of,

x, is given in Fig. 149; the part of this curve, for which, $x < \xi_i$, can be constructed by the following process.

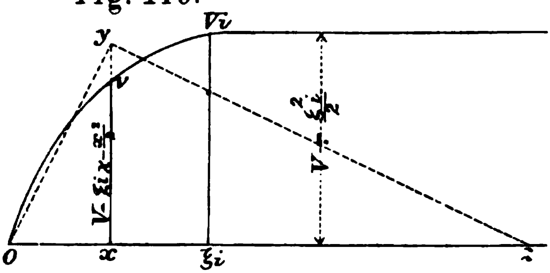
Taking the general equation, expressing the value of, V, namely,

$$V = \xi_i x - \frac{x^2}{2},$$

suppose, $x = o$, and it will be seen that, V, also vanishes; and, when $x =$ the unit of scale,

$$V = \xi_i - \frac{1}{2}.$$

Fig. 149.



Moreover, V, is a maximum when

$$\frac{dV}{dx} = o = \xi_i - x;$$

or, when $x = \xi_i$, which value of x being substituted in the general equation for, V, gives

$$V = \frac{\xi_i^2}{2}. \quad \text{(See Note.)}$$

In the preceding investigation it will have been observed that the loads, $P_1, P_2, \ldots . P_n$ enter lineally; that is, under their first powers, into all the equations. The consequence of this is that the, $n + k + 2$, unknown quantities; namely, the, n, reactions, R; the, k, independent moments, μ; and the two

Note.—In Fig. 149, $Ox =$ unit of scale; $xy = O\xi_i = \xi_i$; $\overline{xy}^2 = \xi_i^2 = Ox.\ xz = xz$; $\overline{\xi_i V_i} = \frac{xz}{2} = \frac{\xi_i^2}{2}$.

constants, y_0 and $\frac{d\,y_0}{d\,x_0}$, will all be expressible as linear functions of the loads, P . , and, as these loads all enter in the same way into the general equations, the bending moment, M, and the shearing force, $F = \frac{d\,M}{d\,x}$, derived from it, will bear values which are due to the sum of the effects of the loads, P, taken separately ; so that, for example, the resultant moment of the system, M, is equal to the sum of the partial moments, deduced on the supposition that each of the forces, P and R, and each of the independent moments, μ, produces an effect of its own distinct from that of the others. This statement contains the enunciation of the *theory of superposition of forces*, which permits us to calculate the general bending moment induced at any section by considering each of the forces and reactions separately, taking the moment of each relatively to the cross-section considered, and finally adding together the partial bending moments thus found.

EXAMPLES.

1. What would be the *maximum* unit-stress, produced in the cross-section, 65 feet from the abutment, A, Ex. 1, Pt. III. Ch. V. assuming the form of section to be a double, T, of the following dimensions ; total depth, h = 8 ft. ; depth of flanges, h_1, = 2·4 inches ; breadth of flanges, b_1 = 1 ft. ; and breadth of web, b = 0·6 inches ?

Maximum Stress = 6·5 tons per sq. in.

2. Taking the same example, what would be the stress, arising from shearing forces, at a section, 64 feet from A, considering the shearing stress to be uniformly distributed over the web ?

Shearing Stress = 82 lbs. per sq. in.

3. Under similar conditions, what would be the *maximum* shearing stress, due to shearing forces, at the same section ?

Maximum Shearing Stress = 123 lbs. per sq. in.

4. What would be the *maximum* stress, induced at the crown-section of the circular arch, considered in Ex. 8, Pt. III. Ch. V. assuming the cross-section to be made of a solid rectangular form, having a total depth at the crown, $h = 2$ ft.; and a uniform breadth, $b = 2{\cdot}4$ inches?

Maximum Stress = 47·5 tons per sq. in.

5. Supposing the solid rectangular section, described in the last example, to be remodelled, by taking away some of the material from the top; that is, cutting away from each side two rectangular parts, each 1·75 ft. long and 0·6 in. broad, and transposing them to the bottom of the section, so as to form a bottom flange; what would be the new dimensions of the reduced upper part or web, and of the flange, assuming that the breadth of the latter is reduced, so as to make the total area of the reformed section 6·25 per cent. less than the area of the old rectangular section?

Area of web = $1{\cdot}75 \times 0{\cdot}1$ sq. feet.
Area of Reduced Flange = $0{\cdot}25 \times 0{\cdot}8$ sq. feet.

6. Find the distance, from the under line of the flange, of the centre of gravity of the improved simple T section just described.

Distance = 0·591 feet.

7. If the cross-section of the arched rib were made after this new model, what would be the *maximum* stress, produced in the extreme fibres of the flange?

Maximum Stress = 26·15 tons per sq. in.

8. Design the arch of steel, according to the same model of cross-section, of the dimensions: web, 2 feet by 1·2 inches; flange, 1 foot by 6 inches; and find the maximum stress in the extreme fibres of the flange.

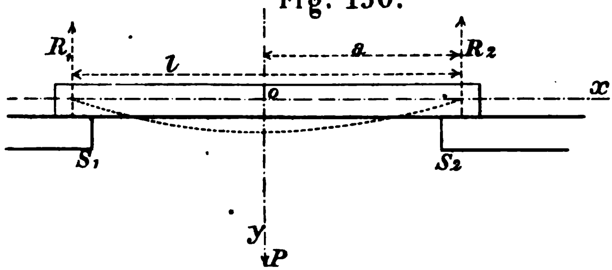
Maximum Stress = 12·4 tons per sq. in.

CHAPTER VII.

SOLID GIRDERS IN EQUILIBRIUM.

1. GIRDERS SUPPORTED AT BOTH ENDS.—Let us suppose a straight girder simply supported at the ends, S_1, S_2, Fig. 150, and bearing a concentrated load, P, at the centre. The re-

Fig. 150.



actions at the supports, due to the load, P, will be each equal to, $\frac{P}{2}$. Taking the origin at the centre point, O, of the girder; the axis of, x, along the neutral fibre of the girder; and that of, y, downwards at right angles to this line, we shall find, according to the principles laid down in the preceding chapter, that the bending moment at any section, distant, x, from, O, will be equal to

$$M = e\,r.^2 \frac{d^2 y}{dx^2} = -\frac{P}{2}\left[\frac{l}{2} - x\right],$$

in which expression, l, represents the span of the girder.

Integrating the above equation on the supposition that the

dimensions of the cross-sections and consequently, $e\,r^2$, are constants, we obtain

$$e\,r^2.\,\frac{dy}{dx} = -\frac{P}{2}\left[\frac{l}{2}x - \frac{x^2}{2}\right],$$

no constant being added; since, when $x = o$, $\frac{dy}{dx} = o$.

By a second integration,

$$e\,r^2.\,y = -\frac{P}{2}\left[\frac{l.\,x^2}{4} - \frac{x^3}{6}\right] + C.$$

To determine, C, we have the condition that, when $x = \frac{l}{2}$, $y = o$, which gives

$$C = \frac{P.\,l^3}{48};$$

so that finally

$$e\,r^2.\,y = \frac{P}{2}\left[\frac{x^3}{6} - \frac{l.\,x^2}{4} + \frac{l^3}{24}\right].$$

The ordinate of deflection, y_o, at the centre of the girder, corresponding to the zero-value of x, is,

$$y_o = \frac{P}{48}.\frac{l^3}{e\,r^2}.$$

Had the load been uniform instead of concentrated, each of the reactions at, S_1 and S_2, would have been equal to half the total distributed load, $p.\,l$; where, p, is the load per unit-length of span.

In this case, the bending moment at any section, distant, x, from, O, could be found by the following process.

Let, x_1, be the abscissa of any point of the neutral fibre, situate between the right extremity of the girder and the plane of section at x. The bending moment, produced at the plane of section by the unit-load, p, at, x_1, is equal to

$$p\,(x_1 - x).$$

Taking the sum of all such moments between the limits, $\frac{l}{2}$

and x, we find the bending moment due to uniform load to be

$$\int_x^{\frac{l}{2}} p\,(x_1 - x)\,dx_1,$$

where, x_1 is supposed variable.

But,

$$\int_x^{\frac{l}{2}} p\,(x_1 - x)\,dx_1 = p\left[\frac{x_1^2}{2} - x\,x_1\right],$$

which, taken between the given limits of, x_1, gives the result,

$$\frac{p}{2}\left[\frac{l^2}{4} + x^2 - lx\right]$$

Again, the bending moment, produced at the same section by the reaction, $p\frac{l}{2}$, at S_2, will be equal to,

$$-p\,\frac{l}{2}\left(\frac{l}{2} - x\right) = \frac{p}{2}\left(l\,x - \frac{l^2}{2}\right);$$

so that, adding the effects due to uniform loads and the reaction at the support, S_2, the complete value of the bending moment, M, at the section, x, is

$$M = \frac{p}{2}\cdot\left[x^2 - \frac{l^2}{4}\right] = e\,r^2\,\frac{d^2y}{dx^2};$$

whence, by a first integration,

$$e\,r^2\,\frac{dy}{dx} = \frac{p}{2}\left[\frac{x^3}{3} - \frac{l^2.\,x}{4}\right],$$

and integrating again,

$$e\,r^2.\,y = \frac{p}{2}\left[\frac{x^4}{12} - \frac{l^2\,x^2}{8}\right] + C.$$

But, when $x = \frac{l}{2}$, $y = o$; therefore,

$$C = \frac{5}{384}\,.\,p.\,l^4;$$

wherefore finally

$$e\,r^2.\,y = \frac{p}{24}\left(\frac{l^2}{4} - x^2\right)\left(\frac{5.\,l^2}{4} - x^2\right).$$

The central ordinate of deflection, corresponding to the zero-value of, x, is

$$y_0 = \frac{5.\,p.\,l^4}{384.\,e\,r^2}\,.$$

If the half-span, $\frac{l}{2}$, be represented by, a, the equations of the deflected curve take a simpler form ; that for a load concentrated at the centre becoming

$$e\,r^2.\,y = \frac{P}{12}\,(a - x)\,[2\,a^2 + 2\,a\,x - x^2],$$

and that for uniform load,

$$e\,r^2.\,y = \frac{p}{24}\,(a^2 - x^2)\,(5\,a^2 - x^2).$$

In the latter case the curve approximates in form to the parabola,

$$y_1 = \frac{5\,p\,a^2}{24\,e\,r^2}\,(a^2 - x^2),$$

obtained by neglecting the term,

$$-\frac{p}{24}\,[a^2 - x^2].\,x^2.$$

The maximum value of this term corresponds to the value of, x, determined by the relation,

$$\frac{d}{dx}\,(a^2 - x^2)\,x^2 = o,$$

which gives,

$$2\,a^2\,x - 4\,x^3 = o,$$

or,

$$x = \pm\,a\,\sqrt{\tfrac{1}{2}}\,.$$

This value of, x, being substituted in the equations for, y, and y_1, furnishes the results,

$$y = \frac{9}{4} \cdot \frac{p\,a^4}{24\,er^2}$$

$$y_1 = \frac{10}{4} \cdot \frac{p\,a^4}{24.\,e\,r^2}$$

Consequently the greatest difference between, y_1 and y, or the value of the neglected term, is,

$$y_1 - y = \frac{1}{4} \cdot \frac{p\,a^4}{24\,e\,r^2},$$

and this error, expressed as a fraction of the correct value of, y, will be,

$$\frac{y_1 - y}{y} = \frac{1}{9};$$

hence the greatest error committed in substituting the parabolic curve for the curve of deflection, as measured by a comparison of the ordinates, is about 11 per cent.

The central ordinate of deflection expressed in terms of the half-span is,

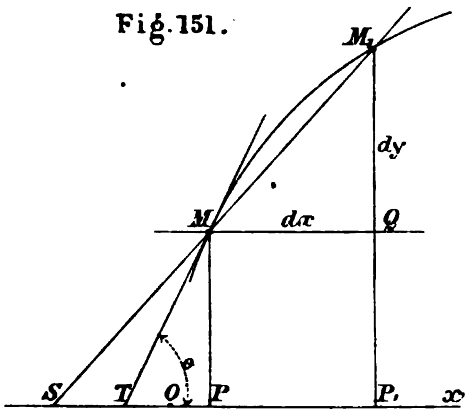
$$y_o = \frac{5}{24} \cdot \frac{p\,a^4}{e\,r^2}.$$

2. THE ANALYSIS OF CURVES.—In applications of analytical methods to mechanical subjects, it is often found necessary to represent the various functions of the variables by a graphic diagram. This is sometimes done by taking lengths along a straight line, $O\,x$, to represent one kind of value, and ordinates, y, perpendicular to the axis of, x, to typify the corresponding values, expressed analytically as functions of the abscissæ, x. It is useful to investigate what are the ordinary relations subsisting between these two classes of values.

Let any axis, $O\,x$, Fig. 151, be chosen along which to set off a series of increasing lengths, representing the increasing values of, x. Perpendicularly to this axis, set off the corresponding values of, y; the upper ends of the series of ordinates, y, will

determine a curve, $M M_1$. Through any two points on this curve, such as, M and M_1, draw a line-secant, $\overline{M_1 M}$, and produce it to meet the axis of, x, in a point, S. At, M, draw a tangent to the curve, meeting the axis of, x, in, T. Let $\overline{MP}$, $\overline{M_1 P_1}$, be the ordinates of the curve corresponding to the abscissæ, $\overline{OP}$, $\overline{OP_1}$, and through, M, draw $\overline{MQ}$, parallel to Ox.

Fig. 151.



From the construction of the figure, it follows that the triangles, $M_1 M Q$ and $M S P$, are similar, and that therefore,

$$\frac{M_1 Q}{M Q} = \frac{M P}{P S}.$$

When the two ordinates, $P M$, $P_1 M_1$, are drawn very closely together so as almost to coincide, $\overline{M_1 Q}$, expressing the difference,

$$\overline{P_1 M_1} - \overline{P M} = y_1 - y,$$

can be equated to, dy, where, dy, represents the difference between two values of, y, very approximately equal to each other, and corresponding to two values of, x, differing by a small value, $\overline{MQ} = dx$. The ratio, $\frac{M_1 Q}{M Q}$, is thus defined by help of the general symbol, $\frac{dy}{dx}$.

But, as the ordinate, $\overline{P_1 M_1}$, approaches, $\overline{P M}$, the secant-line, $\overline{M_1 S}$, gradually approaches the line-tangent, $\overline{M T}$, being in the limit confounded with it. Ultimately, therefore,

$$\frac{P M}{P S} = \frac{M_1 Q}{M Q} = \frac{P M}{P T} = \frac{d y}{d x} = tan. \, M\hat{T}P = tan. \, \theta$$

If the difference, dy, is positive, as shewn in Fig. 151; tan θ, will also be positive; that is, it will increase in value

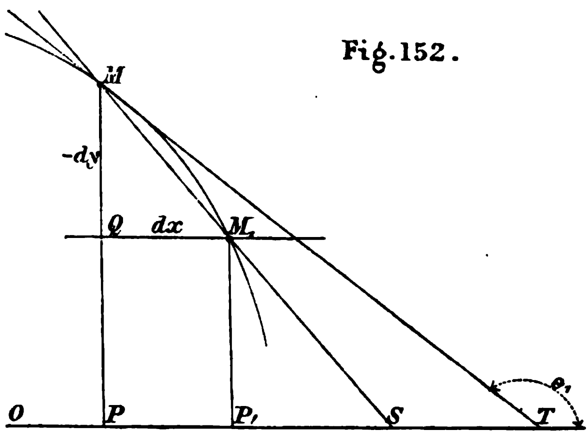


Fig. 152.

simultaneously with the abscissæ, x. But if, dy, represent a decrease in the value of y, corresponding to a value of x increased;—a case illustrated in Fig. 152, where, $\overline{P_1 M_1} - \overline{P M} = - \overline{MQ}, = -dy$;—then,

$$tan. \, \theta_1 = - \frac{dy}{dx}.$$

This relation expresses the fact that the angle, θ_1, is the supplement of, $M\hat{T}P$, the tangent of which is equal in absolute value to, $\frac{dy}{dx}$.

In each case, therefore, it is very necessary to pay attention, not only to the absolute value of, $\frac{dy}{dx}$; but as well to its sign.

From the relation,

$$\frac{dy}{dx} = \frac{P M}{P T},$$

is deduced,

$$PT = y \div \frac{dy}{dx}.$$

If the equation to the curve be known, both, y and $\frac{dy}{dx}$, can be expressed in terms of, x, and in this way the sub-tangent, $\overline{PT}$, corresponding to any point, M, on the curve can be fully determined. The point, T, being known, fixes the direction of the line-tangent, $\overline{MT}$, and by immediate construction that line itself.

If the value of the ratio, $\frac{PM}{PT}$, be given, that of $\frac{dy}{dx}$, which is equal to it, is thereby known, and by integrating the differential equation,

$$\frac{dy}{dx} = \frac{PM}{PT},$$

the law, which connects the co-ordinates, x and y, will be discovered. If the law established between these co-ordinates be given, the curve representative of it may be described, point by point, by setting off a series of values of y, corresponding to arbitrary values of x, and found by a solution of the general equation.

Let, CD, Fig 153, be any curve determined by the principles here explained, and let three consecutive ordinates, $\overline{MQ}$, $\overline{NR}$, and, $\overline{PS}$, be separated by distances, $\overline{QR} = \overline{RS} = h$. Let, $\overline{OQ} = x$; then,

$$\overline{OR} = x + h; \; \overline{OS} = x + 2h.$$

It follows from the construction that the triangles, MNX and TZN, are equal; hence

$$\overline{TZ} = \overline{NX} = \overline{RN} - \overline{MQ} = dy.$$

Moreover,

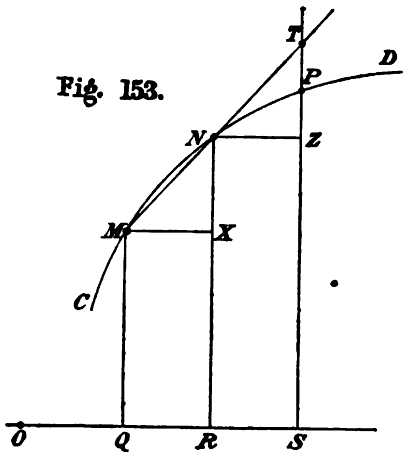
$$\overline{TP} = \overline{TZ} - \overline{PZ} = \overline{NX} - \overline{PZ}.$$

It will be useful to examine the symbolic meaning embodied in the geometric line, $\overline{TP}$.

Since, $\overline{QM}$, is expressed by the symbol, y, and $y = \phi\,(x)$;

where the symbol, ϕ, implies that, y, is some function of, x; $\overline{RN}$, will be the value, $\phi\ (x)$, takes, when $(x + h)$ is substituted for, x.

Fig. 153.



By Taylor's theorem, therefore,

$$\overline{RN} = y_1 = \phi\ (x + h) = \phi\ (x) + \phi'\ (x).\, h + \phi''\ (x)\frac{h^2}{2} + \\ + \phi'''(x)\ \frac{h^3}{3} + \ldots .$$

Again,

$$\overline{PS} = y_2 = \phi\ (x + 2\,h) = \phi\ (x) + \phi'\ (x).\ 2\,h + \phi''\ (x).\frac{(2\,h)^2}{2} + \\ \phi'''\ (x)\ \frac{(2\,h)^3}{3} + \ldots .$$

wherefore,

$$\overline{TZ} = \overline{NX} = \overline{RN} - \overline{MQ} = \phi'\ (x).\ h + \phi''.\ (x).\ \frac{h^2}{2} + \ldots .$$

Further,

$$\overline{PZ} = \overline{PS} - \overline{RN} = \phi'\ (x).\ h + \frac{3}{2}.\ \phi''\ (x).\ h^2 + \ldots .$$

hence, if, $dx = h$,

$$\overline{TP} = \overline{TZ} - \overline{PZ} = -\ \phi''\ (x).\ h^2 = -\ \frac{d^2 y}{dx^2}.\ dx^2 = -\ d^2 y$$

But,

$$\overline{TZ} = \overline{NX} = \overline{RN} - \overline{MQ} = y_1 - y = dy;$$
$$\overline{PZ} = \overline{PS} - \overline{RN} = y_2 - y_1 = dy_1;$$

whencc,

$$\overline{TP} = dy - dy_1 = -[dy_1 - dy] = -d.[dy] = -d^2y;$$

by which equation the meaning attaching to the symbol d^2y is fully determined to be,

$$d[dy] = \pm d^2y = \overline{TP}.$$

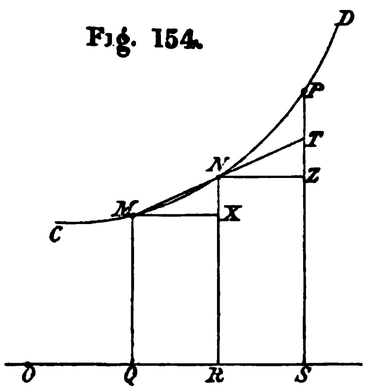
The reason for the double sign being prefixed is this; when the curve is concave towards the axis of x, as shewn in Fig 153,

$$d[dy] = \overline{PZ} - \overline{NX} = \overline{PZ} - \overline{TZ} = -\overline{TP};$$

on the other hand, when the curve is convex towards the same axis, Fig 154,

$$d[dy] = \overline{PZ} - \overline{TZ} = +\overline{TP}.$$

Fig. 154.

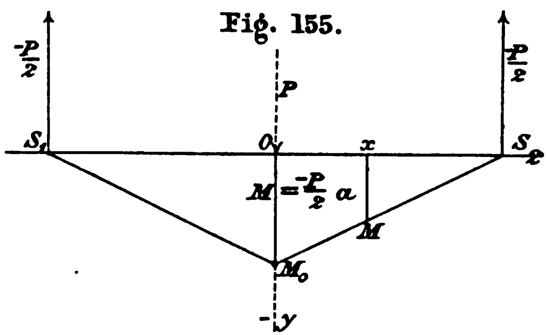


If, however, the curvature, whilst remaining concave or convex towards the axis of, x, fall below that axis; so that, y and dy, *change sign*, the converse of the above statement will be true; for the change of sign will affect all the geometric quantities involved; whilst their absolute values and relations to each

other in space remain the same. In this case, therefore, the curve being situate *below* the axis of, x, and having its concavity towards that axis, $\frac{d^2y}{dx^2}$, will be a positive quantity; and when convex under the same conditions, $\frac{d^2y}{dx}$ will be a negative quantity. Generally speaking, and having regard to all cases, the curve will be concave or convex towards the axis of, x, accordingly as y and $\frac{d^2y}{dx^2}$, are of different or similar signs. It may, for example, happen that ordinates measured in a downward sense are considered positive; and it will be then evident that the remarks, made above with respect to the curve lying above the axis of, x, apply with equal correctness to the curve below that axis, the general rule just given not being affected by the change in the sign of, y.

3. GRAPHIC TREATMENT OF SOLID GIRDERS.—Let the girder be simply supported at each end, Fig. 155, and support

Fig. 155.



a load, P, concentrated at the centre;—the bending moment at any section distant, x, from, O, is found by means of the equation [p. 287],

$$M = -\frac{P}{2}[a - x],$$

in which upward forces and left-handed moments are considered negative.

Construct a curve, or line-limit, having the above equation

for its analytical expression, and referred to axes passing through, O, parallel and perpendicular to the horizontal neutral axis of the girder before deflection.

By the given equation,

$$M = \frac{P}{2} \cdot x - \frac{P}{2} a,$$

which is an equation to a straight line. Now, when $x = a$, $M = O$; or the bending moment vanishes at the ends of the girder. Again, when $x = o$, $M = -\frac{Pa}{2}$. Hence, if along the axis of, y, a length, $OM_o = -\frac{Pa}{2}$, be set off, and the extremity, M_o, be joined to, S_1 and S_2, the triangle, $S_1 M_o S_2$, will form what may be called the *curve of moments*;—that is, any ordinate, xM, drawn from a point, x, to meet the line-limit of the triangle, will give the graphic value of the bending moment produced at that part of the beam.

In the case of a uniformly distributed load the equation of moments takes the form,

$$M = -\tfrac{1}{2} p(a^2 - x^2),$$

where, p, is the load per unit-length of the span ($=2a$). Here the curve of moments is a parabola, the central ordinate to which is,

$$M = -\tfrac{1}{2} pa^2,$$

and the end ordinates, corresponding to the abscissæ, $x = a$, are seen to vanish.

Hence, to describe the curve, set off along the axis of, y, a distance, $O M_o = -\tfrac{1}{2} pa^2$, [Fig. 156]. Complete the rectangle, $O M_o D S_1$; and divide the lines, $S_1 O$ and $S_1 D$, into the same number of parts, commencing in each case at the point S_1. From the points of division of the line, $S_1 O$, draw lines parallel to $S_1 D$, and from the points of division on, $S_1 D$, draw lines to the centre, M_o. The points, determined by the intersection of lines similarly marked, will give any number of points on the parabolic curve required. Any intercept, xM,

will then give the graphic value of the bending moment acting at the section through, x.

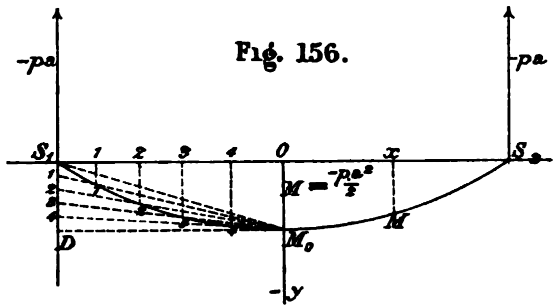
Since, $\frac{1}{2} p.a^2$, is always greater than, $\frac{1}{2} px^2$, whenever, M, exists, it follows from the equation,

$$M = -\tfrac{1}{2} p\,(a^2 - x^2),$$

that, M, is always negative between the limits, a and o. Moreover,

$$\frac{d^2 M}{d\,x^2} = p,$$

Fig. 156.



from which relation it may be inferred that, $\frac{d^2 M}{dx^2}$ is always positive between the same limits.

Wherefore, by a previous rule [Ch. VII. § 2] since M and $\frac{d^2 M}{dx^2}$, are of different signs the curve is concave towards the axis of, x.

Let, F, be the general symbol for the shearing force at any section; then,

$$F = \frac{d\,M}{dx} = \frac{P}{2};\text{—for concentrated load.}$$

$$F = \frac{d\,M}{dx} = p\,x;\text{—for uniform load.}$$

These are the true *absolute* values of, F, in the two cases considered; but, since it has been assumed that negative forces, such as, $-\frac{P}{2}$ give rise to left-handed moments; or, in

other terms, forces of any particular sign produce moments of the same sign; it follows [Pt. III. Ch. VI. § 3] that, F and $\frac{d\,M}{d\,x}$, are opposite in sign; consequently,

$F = -\frac{P}{2}$;—for load concentrated at the centre of girder

$F = -p.x$;—for uniform load.

As regards the maximum deflection occurring at the centre of the span, it will be seen [Pt. III. Ch. VII. § 1] that, if, y_o, represent the maximum deflection for the first case, when the load, P, is concentrated at the centre of the girder,

$$y_o = \frac{P.a^3}{6.e\,r^2}.$$

Similarly, for uniform load, the maximum central deflection of the girder may be represented by,

$$y_o' = \frac{5}{24}.\frac{p\,a^4}{e\,r^2}.$$

Let us suppose it were required to find the particular load, P, concentrated at the centre of span, which would produce the maximum deflection, y_o', due to the uniform load, p, per unit-length of girder. In this instance, it is necessary that,

$$y_o = y_o';$$

or,

$$\frac{P\,a^3}{6\,e\,r^2} = \frac{5}{24}.\frac{p\,a^4}{e\,r^2},$$

which leads to the relation,

$$P = \frac{5}{4}\,p\,a = \frac{5}{8}\,p\,l,$$

where, l, is the entire length of the span. Hence, a concentrated load equal to $\frac{5}{8}$ths the entire uniform load, $p\,l$, would produce the same central deflection as that due to the load, $p\,l$, uniformly distributed.

If, again, it were required that the maximum bending moments, due to given load, P, concentrated at the centre of the span, and to the uniform load, p, should be equal to each other, we must equate the maximum values of the expressions,

$$M = -\frac{P}{2}[a - x], \text{ and, } M = -\frac{1}{2}p[a^2 - x^2].$$

that is, we must have

$$-\frac{P}{2}.a = -p\frac{a^2}{2},$$

or,

$$P = pa = \frac{pl}{2},$$

which signifies that a concentrated load equal to only *one half* the uniformly distributed load, pl, would produce the same bending moment at the centre of the girder. Since, moreover, the greatest longitudinal stress at any section of a girder of given transverse dimensions is equal to,

$$t = \frac{M}{I}.\frac{h}{2},$$

it may be concluded that the maximum longitudinal stresses obey the same law as the maximum bending moments; and therefore the same maximum stress will be produced by a load, P, concentrated at the centre of the girder, as would be created by a load, $2P$, equably distributed over the length of the span.

In the case where there is both a concentrated and uniform load applied to the girder, the total effect produced by the two loads will be equal to the sum of the effects of the loads separately taken [Pt. III. Ch. VI. § 5]. Therefore, the bending moment due to a load, P, concentrated at the centre, and a uniformly distributed load, pl, will be equal to

$$M = -\left[\frac{P}{2}(a - x) + \frac{p}{2}(a^2 - x^2)\right],$$

in which expression, $a = \frac{l}{2}$.

To take a more general case, it may be supposed that the load, P, is applied anywhere along the length of the girder, say at a fixed distance, x_1, from the support, S_2, Fig. 157, which is here taken as the origin. The reactions at, S_1 and S_2, can be then found by means of the relations,

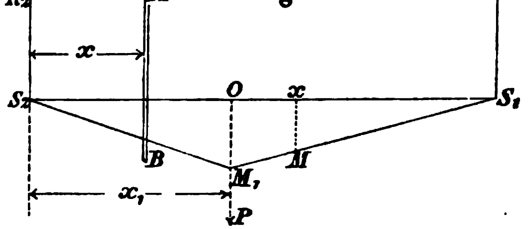
$$P : R_1 : R_2 :: l : x_1 : \overline{l - x_1} \qquad \text{[Fig. 85.]}$$

from which are deduced the values of, R_1 and R_2; namely,

$$R_1 = \frac{x_1}{l} . P; \; R_2 = \frac{\overline{l - x_1}}{l} . P.$$

The bending moment produced at a section, $\overline{AB}$, of the girder, distant, x, from the origin, S_2, will depend for its magni-

Fig. 157.



tude on the variable abscissæ, x. Suppose the section to lie on the near side of the load; so that, $x < x_1$; then,

$$M = R_2 . x = \frac{P . x}{l} [l - x_1],$$

in which equation right-handed moments have been made positive, and the limits of moments have been taken between the plane of section and the left-hand extremity of the beam. If these limits had been assumed between the sectional plane and the right-hand extremity of the girder, we should have obtained the same *absolute* value for, M, namely,

$$\begin{aligned} M' &= - R_1 [l - x] + P [x_1 - x] \\ &= - \frac{P x_1}{l} [l - x] + P [x_1 - x] = - \frac{P x}{l} [l - x_1], \end{aligned}$$

where again right-handed moments are considered positive.

The two moments, M and M', being equal and opposite, balance each other, and maintain the section, $A\,B$, in equilibrium.

Secondly, suppose the plane of section to lie on the side of the load, P, removed from the origin; so that, $x > x_1$; then the moment will be expressed by,

$$M = R_1 . [l - x] = \frac{P x_1}{l} . [l - x].$$

Now, imagine the load, P, constantly applied at the same point, x_1; whilst the plane of section, defined by the abscissæ, x, varies in position along the length of the girder. Let it be required to find the section of rupture of the girder under the condition of fixed load. This can be done by determining the value of, x, corresponding to the maximum of, M, in the comparison of the expressions,

$$M = \frac{P x}{l} . [l - x_1],$$

the equation of moments, when, $x < x_1$, and

$$M = \frac{P x_1}{l} . [l - x],$$

the equation of moments, when $x > x_1$.

Comparing these two expressions for, M, it will be seen that in the first equation, the value of, M, increases; whereas in the second form it decreases with the increase of, x. Hence, in the first equation the maximum of, M, corresponds to the greatest admissible value of, x; namely, $x = x_1$;—and in the second equation the maximum of, M, is determined by giving to, x, its least admissible value, which is again, $x = x_1$.

It is thereby made clear that the greatest bending moment, produced by a fixed load, P, takes place at the section, where the load is applied; and that, if the abscissa of the point of application be, $x = x_1$, the absolute value of this bending moment will be,

$$M = \frac{P x_1}{l} . [l - x_1].$$

Consequently, if on the line of load, $\overline{OP}$, Fig. 157, there be set off a length,

$$\overline{O}M_1 = M = \frac{P x_1}{l} . [l - x_1],$$

and the point, M_1, thus found, be joined by straight lines to the points of support, S_1 and S_2, the triangle, $S_1 M_1 S_2$, will form the curve of moments corresponding to a fixed load, P, at x_1. Any intercept, $x M$, drawn in the space enclosed by this triangle, will determine the graphic value of the bending moment at, x.

As a further extension of the above case, let the load, P, travel along the length of the girder; or in other terms let the abscissæ, x_1, of its point of application be considered variable. It is, then, obvious, that the expression,

$$M = \frac{P x_1}{l} [l - x_1],$$

must bear a maximum value corresponding to some particular value of the variable, x_1,which can be found by a solution of the equation,

$$\frac{d M}{d x_1} = o = P [1 - \frac{2 x_1}{l}],$$

by which, $x_1 = \frac{l}{2}$.

Hence, the maximum of the ordinates, $O\ M_1$, or in other words the *maximum maximorum* of the expression,

$$M = \frac{P x}{l} [l - x_1],$$

occurs, when the load, P, is applied at the centre of the girder.

Substituting, $x_1 = \frac{l}{2}$, in the equation,

$$M = \frac{P . x_1}{l} [l - x_1],$$

we find that at the centre of span,

$$M = \frac{P . l}{4} = \frac{P}{2} . a.$$

The general equation to the curve limiting the ordinate-maxima, $O\,M_1$,

$$\overline{O\,M_1} = \frac{P.x_1}{l}\,[l - x_1]$$

$$= P\,x_1 - \frac{P.x_1^2}{l},$$

that is,

$$-\,\overline{O\,M_1} = \frac{P\,x_1^2}{l} - P\,x_1,$$

can be transformed by making, $x_1 = x + \frac{l}{2}$, which is equivalent to removing the origin from, S_2, to the centre of the girder. After this change we shall have

$$-\,O\,M_1 = \frac{P}{l}\left[\left(x + \frac{l}{2}\right)^2\right] - P\left[x + \frac{l}{2}\right]$$

$$= \frac{P}{l}\left[x^2 - \frac{l^2}{4}\right].$$

The ordinate, $\overline{O\,M_1}$, will be seen to vary as, x^2, and a curve formed by setting off a series of values, $O\,M_1$, referred to particular values of, x, will determine a parabolic curve, which may be termed the curve of *moment-maxima*, corresponding to different points of application of the moving load, P. At the centre of span; that is, when, $x = o$,

$$+\,\overline{O\,M_1} = \frac{-\,P.l}{4} = \frac{-\,P\,a}{2} \qquad \text{[Fig. 158.]}$$

Consequently, if we draw a parabola, similar in construction to that shewn in Fig. 156, having for the abscissa of its vertex, [Fig. 158],

$$\overline{O\,M_o} = \frac{-\,P\,a}{2},$$

and for double ordinate,

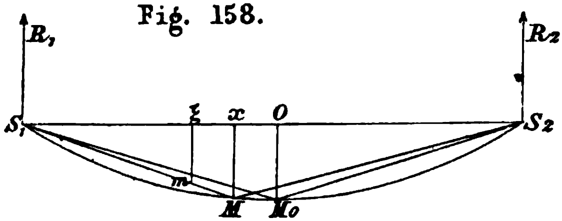
$$\overline{S_1\,S_2} = l,$$

any intercept, $x\,M$, drawn within the limits of this curve and the horizontal line, $\overline{S_1\,S_2}$, will define the graphic value of the

greatest bending moment due to a load, P, applied at, x. It has previously been shewn that the curve of moments for a load, P, applied at, x, consists of two straight lines, drawn from the extremity, M, to meet the points, S_1 and S_2. The same law applies to any point on the curve of maxima, $S_1\ M\ M_0\ S_2$.

In this way we can trace out by a very simple process the various species of bending moments induced along the

Fig. 158.



girder by a load, P, varying in position. For this purpose it is only necessary to construct the parabolic curve of moment-maxima, with a central ordinate,

$$O\,M_0 = \frac{-P\,a}{2}.$$

Then, any ordinate of this curve will give the maximum bending moment for the point of application to which it is drawn; and if straight lines be set off from the end, M, of this ordinate to the points of support, S_1, S_2, any secondary ordinate, $\xi\,m$, drawn from any point, ξ, on the centre line of girder, to a point, m, on the line-limit, $S_1\,M$, of the corresponding triangle of moments, will give the graphic value of the bending moment produced at, ξ, by reason of a concentrated load applied at, x.

It has been proved [p. 298] that if a girder be subject to a uniformly distributed load, the curve of moments will assume the form of a parabola, having a central ordinate,

$$O\,M_0 = -\tfrac{1}{2}\,p\,a^2 \qquad \text{[Fig. 156.]}$$

In order, therefore, that this parabola may become identical with that just drawn to represent the *maxima maximorum* of

the moments due to the load, P, applied at various points on the girder, it is necessary to equate the values of the central ordinates of the two curves ; so that

$$\frac{P\,a}{2} = \tfrac{1}{2}.\,p\,a^2,$$

or,

$$P = p\,a = \frac{p\,l}{2};$$

wherefore, it may be concluded that a moving load, P, equal to *one half* the uniformly distributed load, $p\,l$, produces the same bending moments, and consequently the same longitudinal or fibral stresses, as the uniform load, $p\,l$, at sections of the girder instantaneously under the action of the variable load.

In the preceding investigation the cross-section of the girder has been assumed to be of constant dimensions ; but in the case of beams of uniform strength [Pt. III. Ch. VI. § 2], this law does not obtain. Here the cross-section of the girder varies from one point to another along the line of girder.

It will be useful, therefore, to remember that by changing the area of cross-section according to any varying law, the absolute values of the shearing forces and bending moments are not thereby affected ;—for the reason, that they are functions, not of the area of cross-section, but of the applied loads and the span of the bridge. Still, a change in the area of cross-section will alter the nature of the longitudinal stresses, seeing that these stresses are functions of the different moments of inertia of the sections, taking the type functional form,

$$\frac{t}{\omega} = \frac{M.\,y}{I}.$$

Earlier in this work [Pt. III. Ch. VI. § 2], it has been shewn how these longitudinal stresses can be kept constant by varying one of the transverse dimensions of the girder.

When the area of cross-section varies, the term,

$$e.\,r^2 = \Sigma.\,E.\,\omega.\,y^2,$$

where $e = \Sigma . E \omega$, also varies, and the integration of the differential equation,

$$e r^2 . \frac{d^2 y}{d x^2} = M,$$

becomes in consequence more difficult and complicated. Taking, for example, the case of a rectangular section of the depth, h, and breadth, b, let us suppose that the maximum stress per unit of area of cross-section, $\frac{t}{\omega} = \frac{M y}{I}$, remains constant. If the load, P, be concentrated at the centre of span,

$$M = \frac{-P}{2} [a - x]\,;\ I = \frac{b h^3}{12}\,;$$

$$y = maximum = \frac{h}{2}\,;$$

hence,

$$\frac{M . y}{I} = max^m\ cons^t = \frac{-3 P [a - x]}{b h^2} = S\,;$$

but, $3 P$ and b, are constant, and, h, variable; wherefore, the above condition reduces to the simpler form,

$$\frac{(a - x)}{h^2} = C,$$

where, C, represents the maximum stress, S_o, at the centre of span multiplied by, $\frac{-b}{3 P}$.

Putting $x = o$, in the general value of the stress, we obtain

$$\frac{M_o y_o}{I_o} = S_o = \frac{-3 P}{b} . \frac{a}{h_o^2}\,;$$

whence,

$$C = \frac{a}{h_o^2},$$

and substituting this value of, C, we obtain,

$$\frac{[a - x]}{h^2} = \frac{a}{h_o^2}.$$

Now, since, $e.r^2 = \Sigma . E \,\omega . y^2 = \iint E . y^2 \, dx \, dy$, it is clear that,

$$e . r^2 \propto y^3 \text{ or } h^3,$$

consequently,

$$\frac{e . r^2}{e_o . r_o^2} = \frac{h^3}{h_o^3} = \left[\frac{a - x}{a}\right]^{\frac{3}{2}}.$$

The equation of moments thus becomes,

$$e . r^2 . \frac{d^2 y}{d x^2} = e_o r_o^2 \left[\frac{a - x}{a}\right]^{\frac{3}{2}} . \frac{d^2 y}{d x^2} = -\frac{P}{2} [a - x].$$

Integrating this equation and remembering that, when, $x = o$, $\dfrac{dy}{dx} = o$,

$$e_o r_o^2 . \frac{dy}{dx} = P . a^{\frac{3}{2}} . [a - x]^{\frac{1}{2}} - P . a^2.$$

By a second integration, remembering that, when, $x = a$, $y = o$,

$$e_o . r_o^2 . y = -\frac{2}{3} . P a^{\frac{3}{2}} [a - x]^{\frac{3}{2}} - P . a^2 x + P . a^3.$$

In this equation, put $x = o$, then, y_o, being the ordinate of deflection at the centre of span,

$$y_o = \frac{1}{3 . e_o r_o^2} . P a^3,$$

which is double the value of the same ordinate, when the girder is made of constant section throughout, the area of section being proportioned to the maximum longitudinal stress induced in the beam [Pt. III. Ch. VII. § 1].

If, again, the load, instead of being concentrated, were uniformly distributed and equal to, $p \, l$ or $2 p \, a$, the expression for the bending moment would take the form,

$$M = -\frac{1}{2} . p [a^2 - x^2],$$

and the value of the stress would become,

$$\frac{t}{\omega} = \frac{M.y}{I} = \frac{-\frac{1}{2}p\,(a^2 - x^2)\,\frac{h}{2}}{\frac{b\,h^3}{12}}$$

$$= -\frac{3.p\,[a^2 - x^2]}{b.\,h^2} = S.$$

But, $3\,p$ and b, are constants; therefore

$$\frac{(a^2 - x^2)}{h^2} = C,$$

where, $C = \frac{-b}{3\,p}\ S_o$, and, $S_o = \frac{-3\,p\,a^2}{b.h_o^2}$, whence by substitution,

$$\frac{a^2 - x^2}{h^2} = \frac{a^2}{h_o^2}\,.$$

Moreover,

$$\frac{e.\,r^2}{e_o\,r_o^2} = \frac{h^3}{h_o^3} = \left[\frac{a^2 - x^2}{a^2}\right]^{\frac{3}{2}},$$

and

$$e.\,r.^2\frac{d^2y}{dx^2} = e_o.\,r_o^2\left[\frac{a^2 - x^2}{a^2}\right]^{\frac{3}{2}}.\frac{d^2y}{dx^2} = \frac{-p}{2}.\,[a^2 - x^2],$$

consequently,

$$e_o.\,r_o^2.\,\frac{d^2y}{dx^2} = \frac{-p\,a^3}{2}.[a^2 - x^2]^{-\frac{1}{2}}.$$

By a first integration,

$$e_o.\,r_o^2.\,\frac{dy}{dx} = \frac{-p\,a^3}{2}.\,sin.^{-1}\frac{x}{a}\,.$$

Integrating a second time, and making,

$$\int sin.^{-1}\frac{x}{a}.\,d\,x = \int u\,d\,v,$$

where, $u = sin.^{-1}\frac{x}{a}$, $v = x$, we obtain

$$e_o\,r_o^2\,y = \frac{-p\,a^3}{2}.\left[x\,sin.^{-1}\frac{x}{a} + \sqrt{a^2 - x^2}\right] + C.$$

But, when $x = a$, $y = o$; therefore, finally,

$$y = \frac{p\,a^4}{2.\,e_o\,r_o^2}.\left[\frac{\pi}{2} - \frac{x}{a}.\,sin.^{-1}\frac{x}{a} - \sqrt{1 - \frac{x^2}{a^2}}\right].$$

The value of, y, when $x = o$, is,

$$y_o = \left(\frac{\pi}{2} - 1\right)\frac{p\,a^4}{2\,e_o\,r_o^2}.$$

Comparing this value of, y_o, with that of the same ordinate, when the section is of constant dimensions, the ratio of the two values of, y_o, will be seen to be [Pt. III. Ch. VII. § 1],

$$\left(\frac{\pi}{2} - 1\right)\frac{p\,a^4}{2.\,e_o\,r_o^2} \div \frac{5}{24}.\frac{p\,a^4}{e_o\,r_o^2},$$

or very approximately as, 1·37 to 1.

As a second example illustrating the way to find the nature of the curve assumed by the neutral fibre, when the dimensions of cross-section vary, let us suppose the depth, h, constant, and allow the breadth, b, to change so as to preserve the invariability of longitudinal stress. In this case, the beam being of rectangular cross-section,

$$\frac{t}{\omega} = \frac{M}{I}.\frac{h}{2} = \text{a constant.}$$

Further, supposing the coefficient of elasticity, E, to be constant, we have,

$$E\,I = e.\,r^2 = E.\,\Sigma.\,y.^2\,dxdy,$$

by which,

$$I = \frac{e.\,r^2}{E}.$$

Substituting this value of, I, in the constant expression for longitudinal stress, $\frac{M}{I}.\frac{h}{2}$, there results,

$$\frac{E.\,M}{2.\,e\,r^2}.\,h = \text{a constant;}$$

or since the factor, $\frac{E.h}{2}$, is necessarily the same for all sections,

$$\frac{M}{e\,r^2} = \text{a constant.}$$

But,

$$\frac{M}{e\,r^2} = \frac{M}{E\,I} = \frac{\theta}{L} = \frac{1}{r}. \qquad \text{[Ch. VI. § 1]}$$

Hence, in this particular case, the curvature, $\left[\frac{1}{r}\right]$, assumed by the neutral fibre of the deflected beam, forms an arc of a circle.

Take the case of a load, P, concentrated at the centre of span. If, M_o, be the bending-moment, and, $e.\,r_o^2 = E\,I_o$, the moment of inflexibility at the centre of span,

$$\frac{M}{E\,I} = \frac{M}{e.\,r^2} = \text{a constant} =$$

$$= \frac{M_o}{E\,I_o} = \frac{-\frac{P}{2}\,a}{E\,I_o}$$

$$= \frac{d^2\,y}{dx}.$$

Integrating, we find,

$$\frac{dy}{dx} = \frac{-\,P.\,a\,x}{2.\,E.\,I_o};$$

and, since when, $x = a$, $y = o$,

$$y = \frac{-\,P\,a}{4.\,E.\,I_o}.\,[x^2 - a^2].$$

The value of, y, at the centre of span, will be found by putting, $x = o$, in this equation; that is,

$$y_o = \frac{P.\,a^3}{4.\,E\,I_o};\text{—}$$

the value of the same ordinate, when the section is made constant throughout, is [Pt. III. Ch. VII. § 1]

$$y_0 = \frac{P\,a^3}{6.E\,I_0},$$

so that the ratio of the two deflections is as, 3, is to, 2.

In the case of uniform load, it will be found that,

$$y_0 = \frac{p\,a^4}{4.E.I_0},$$

and the ratio between this value of, y_0, and that for constant section is [Pt. III. Ch. VII. § 1], as 6 to 5.

With respect to the general form of the equation above given for the deflection of the neutral fibre in the present instance, viz.,

$$y = \frac{P\,a}{4\,E\,I_0} \cdot [x^2 - a^2],$$

it should be remarked that, contrary to what has been already stated concerning the constancy of the curvature, this equation represents a parabolic curve. The discrepancy arises from the fact that the demonstration of the equation of the deflected fibre of the beam depends upon the relation,

$$\frac{d^2y}{dx^2} = \frac{M}{E\,I},$$

which does not admit of rigorous proof [Pt. III. Ch. VI. § 1]. Practically the parabolic arc will be almost identical with the arc of a circle.

Let the transverse dimensions of any girder be calculated on the supposition, that the areas of cross-section vary so as to keep the unit of longitudinal stress constant. The deflection, produced by any applied load, will in this case be greater than would take place under the same load applied to a girder of constant section, calculated so as to resist the maximum stress per unit-area of sectional surface. It is,

however, manifest that in the beam of varying section there must exist one particular cross-section of such dimensions that, if the girder were made of that section throughout, it would suffer the same deflection as the beam actually does when made of varying section. Let, x_1, be the abscissa of this section, and, $e_1 r_1^2 = E I_1$, its moment of inflexibility.

In a former article [Pt. III. Ch. VII. § 1] it was shewn that a beam, constructed so as to have a constant moment of inflexibility, $e_1 r_1^2 = E I_1$, and supporting a concentrated load, P, at the middle of the span, undergoes a central deflection represented by,

$$y_o = \frac{P.\,a^3}{6.\,e_1 r_1^2}.$$

This expression shews that the deflections are inversely as the moments of inflexibility. The same principle may be deduced from the general form of the equation [p. 279].

Now, if the moment of inflexibility, $e_o r_o^2$, occurring at the centre of the beam of uniform resistance, were taken as the constant moment of inflexibility throughout the entire length of the girder, only *half* the deflection, which obtains at mid-span of the girder constructed for uniform strength, would be produced in the girder transformed in the way mentioned [p. 309].

Consequently, if the *double* deflection due to the fact that the beam has been made of varying section be preserved, and the sections be made of uniform transverse dimensions with a constant moment of inflexibility, $e_1 r_1^2$, the following relations will hold in the different cases of uniform strength already considered :—

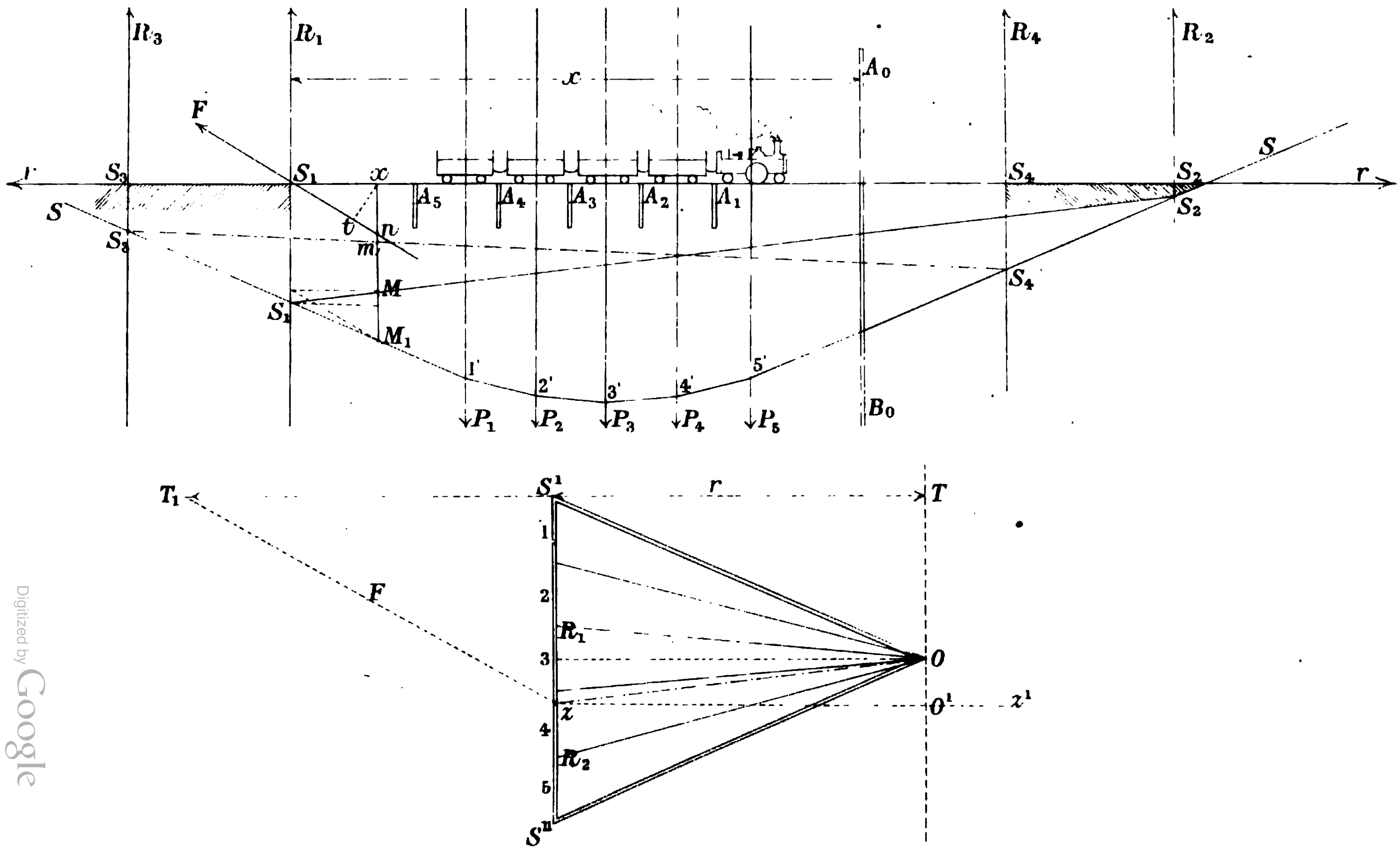
1°. When the depth, h, of the beam varies and the load is concentrated,

$$\frac{e_1 r_1^2}{e_o. r_o^2} = \frac{\frac{1}{2}}{1} = \left[\frac{a - x_1}{a}\right]^{\frac{3}{2}},$$

from which is deduced

$$\frac{x_1}{a} = 1 - \left(\frac{1}{2}\right)^{\frac{2}{3}} = 0{\cdot}370.$$

Fig. 159.



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2°. When the depth, h, varies, and the load is uniform,

$$\frac{e_1 r_1^2}{e_o . r_o^2} = \frac{5}{6(\pi - 2)} = \left[\frac{a^2 - x_1^2}{a^2}\right]^{\frac{3}{2}},$$

whence,

$$\frac{x_1}{a} = \sqrt{1 - \left(\frac{5}{6\pi - 12}\right)^{\frac{2}{3}}}$$

$$= 0{\cdot}435$$

3°. When the depth of the beam is constant, and the breadth varies, for a concentrated load,

$$\frac{e_1 r_1^2}{e_o r_o^2} = \frac{2}{3} = \frac{a - x_1}{a};$$

whence

$$\frac{x_1}{a} = \frac{1}{3}.$$

4°. When the breadth varies, and the load is uniform,

$$\frac{e_1 r_1^2}{e_o r_o^2} = \frac{a^2 - x_1^2}{a^2} = \frac{5}{6},$$

whence,

$$\frac{x_1}{a} = \sqrt{1 - \frac{5}{6}} = 0{\cdot}408.$$

4. Solid Girders Under Moving Loads:—Let there be given a number of forces, P_1, P_2, ... P_5, Fig. 159, applied to a girder in different objective paths. These forces may be taken to represent the loads, containing and contained, of the trucks and engine of a passing train. Let the support, S_1, be chosen as origin, and let the instantaneous abscissæ of the points of application of the loads be represented by, x_1, x_2, ..., x_5. Let the span of the girder, $\overline{S_1 S_2}$ equal, l.

Taking moments about the origin, S_1, and considering left-handed moments positive, we obtain,

$$R_2 . l - [P_1 x_1 + P_2 x_2 + P_3 x_3 + P_4 x_4 + P_5 x_5] \qquad (1)$$

Again, stating the condition of equilibrium that the sum of

the reactions at the supports should equal the sum of the downward forces, or loads,

$$R_1 + R_2 = P_1 + P_2 + P_3 + P_4 + P_5. \qquad (2)$$

By the first equation it is evident that,

$$R_2 < [P_1 + P_2 + P_3 + P_4 + P_5];$$

since each of the factors, $x_1, x_2, \ldots, x_5$, is less than, l; therefore it may be concluded that in any given system, $R_2 < \Sigma. P$.

It can then be inferred from the second equation that, $R_1 < \Sigma. P$.

The absolute values of, R_1, R_2, which are unknown, can be determined by means of the two conditions of equilibrium, (1) and (2).

Suppose a movable plane of section, $\overline{A_0 B_0}$, Fig. 159, to start from a point situate to the right of the series of forces, P, and to be displaced gradually in the direction of the origin, S_1, thus leaving the forces, P, one by one on its right. Bending moments, taken between the limits of the plane, $\overline{A_0 B_0}$, and the right-hand extremity of the girder, corresponding to six different positions, $A_0, A_1, A_2, \ldots, A_5$, will be

$$M = R_2 [l - x]$$
$$M = R_2 [l - x] - P_5 [x_5 - x]$$
$$M = R_2 [l - x] - P_5 [x_5 - x] - P_4 [x_4 - x].$$
$$M = R_2 [l - x] - P_5 [x_5 - x] - P_4 [x_4 - x], \text{ \&c.}$$

The fifth moment will be of similar form, and the sixth, relatively to the position of the plane, A_5,

$$M = R_2 [l - x] - P_5 [x_5 - x] - P_4 [x_4 - x] - P_3 [x_3 - x] - \\ - P_2 [x_2 - x] - P_1 [x_1 - x],$$

in which expression, x, represents the abscissa of the plane at, A_5.

It will be seen that the above values of, M, are all linear functions of, x, the variable abscissa of the moving plane, and each of these equations must, therefore, represent a straight line.

The first equation is the equation of moments for any position of the moving plane between the limits, l and x_5;—the second equation replaces the first between the limits, x_5 and x_4;—the third obtains from, $x = x_4$ to, $x = x_3$; – the fourth between, x_3 and x_2; the fifth from, x_2 to x_1; and the sixth obtains over the range limited by, x_1 and o.

When, $x = l$, the first equation gives, $M = 0$;—similarly, when, $x = o$, the last equation reduces to

$$M = R_2\, l - \Sigma\, P\, x,$$

and, therefore, by the first equation of equilibrium, M, vanishes.

The amounts of the separate loads, P, being supposed known, draw a vertical line, $S'\,S''$, called the *line of forces*, and divide this line into five parts, 1, 2, 3, 4, 5, representing according to an arbitrary scale the series of loads, P.

Choose any pole, O, and draw polar lines to the extremities, $S'\,S''$, and the several divisions on the line., $S'\,S''$.

Construct the polar polygon, S, 1′, 2′, 3′, 4′, 5′, S, leaving the lines, $1'S$ and $5'S$, indeterminate.

The lines of action of the reactions, R_1 and R_2, will intersect the indefinite lines, $1'S$ and $5'S$, in points, S_1 and S_2; and the polar polygon can then be completed by drawing the line, $S_1\,S_2$.

Suppose the train to be at a stand-still in the position indicated.

From any point, x, situate on the girder, draw a vertical line, xM_1, determining an intercept, MM_1, in the polar polygon; then [Pt. III. Ch. V. § 3], the bending moment at the section through, x, produced by the train in its actual position will be

$$M = r\,.\,\overline{MM_1},$$

where, $r = \overline{T\,S'}$, on the polygon of forces. The principle can be arrived at in another manner.

Suppose two equal and opposite forces, r, applied at, S_1 and S_2, respectively, their lines of action being horizontal. These

forces will be independently balanced, and cannot therefore disturb the existing state of equilibrium of the system.

Now, the resultant of the forces acting between the section at, x, and the left-hand support, S_1, will be equal to the resultant of the forces, R_1 and r. This resultant, F, can be shewn graphically on the polygon of forces; for, if from the point, S', the upper extremity of the line representing the reaction, R_1, on the polygon of forces, a line, $S' T_1 = S'T = r$, be drawn;—then the line, T_1z, joining, T_1, with z, the lower end of the graphic term, $S'z = R_1$, will represent the required resultant, F, in magnitude and direction.

Next, through the point of support, S_1, draw a line, $\overline{S_1F}$, parallel to, T_1z. This line, $\overline{S_1F}$, will define the actual path of the resultant force, acting upon the section at, x.

Let the line of action, $\overline{S_1F}$, meet the vertical line, xM_1, in a point, n;—then, supposing the force, F, to act at, n, resolve it into vertical and horizontal components, R_1 and r. The vertical component acts in a line passing through, x, and its moment about that point will, therefore, vanish.

The moment produced at, x, by the action of the horizontal component, r, will be

$$r.\overline{x n}.$$

It will be immediately seen that,

$$\overline{x n} = \overline{M M_1}:$$

for in reality, $\overline{x n}$, is the intercept with respect to the point, x, on a particular polar polygon, constructed relatively to a pole, O', situate at the intersection of the vertical line, $O O'$, with the horizontal line, zz'; and it has been already proved [Pt. III. Ch. V. § 3] that corresponding intercepts, taken from different polar polygons of the same system of forces, are equal and agree in determining the same absolute value of the moment at the point for which they are drawn.

According to the usual definition the moment in the above instance would be represented by the product of the resultant force, F, and the *arm*, xt, Fig. 159, that is by

$$M = F.\overline{x t}.$$

Hence,

$$x\,t = \frac{M}{F} = \frac{r\,.\,\overline{M\,M_1}}{F} = \frac{r\,.\,\overline{M\,M_1}}{T_1\,z}\,.$$

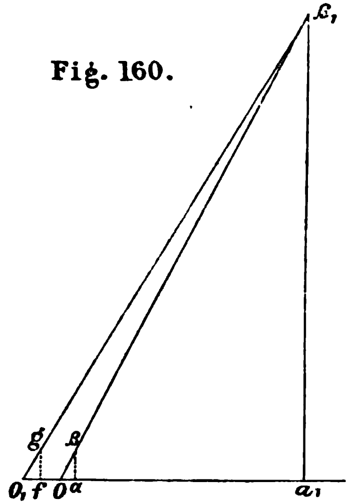
The quotient,

$$x\,t = \frac{r\,.\,M\,M_1}{T_1 z}\,,$$

can be determined by the graphic process shewn in, Fig. 160, where,

$$O\,a = 1\,;\;O\,a_1 = r\,;\;a\beta = \overline{M\,M_1},$$

Fig. 160.



and a perpendicular line erected at, a_1, intersects the line, $a\beta$, produced in a point, β_1, such that,

$$\frac{a_1\,\beta_1}{a\,\beta} = \frac{O\,a_1}{O\,a} = Oa_1$$

or,

$$a_1\,\beta_1 = O\,a_1 .\,a\,\beta = r .\,\overline{M\,M_1}\,.$$

Again, in the same figure, make

$$a_1\,O_1 = T_1 z\,;\;O_1 f = O\,a = \text{unity}.$$

Join, β_1 and O_1, and erect a perpendicular at f, to meet

the line, $O_1\,\beta_1$, in g; then, since by the first part of the process,

$$a_1\,\beta_1 = r\,.\,\overline{M M_1},$$

it follows that,

$$\frac{a_1\,\beta_1}{O_1\,a_1} = \frac{g\,f}{O_1 f} = g\,f,$$

or,

$$g\,f = \frac{r\,.\,\overline{M M_1}}{T_1\,z} = x\,t,$$

which on comparison will be found correct.

It is evident that the polar polygon would change its form, if another system of loads were applied to the bridge; or, again, if the same system of loads were moved forward or backward by a common distance. In the latter case, which embraces that of a train of loaded trucks passing over a bridge, where the distances between the points of application of the loads are equal and constant, and the train looked upon as a whole moves across the girder;—it is easy to arrange the diagram so that one polar polygon will serve for all positions of the load.

Let, P_1, P_2, P_3, P_4, P_5, Fig. 159 represent the concentrated loads of the trucks and engine, regarded as an independent system hanging freely in space. The polar polygon of these forces will correspond to the open figure, S, $1'$, $2'$, $3'$, $4'$, $5'$, S.

Suppose, now, the series of loads, P, to be brought to bear upon the girder of a bridge, having its supports at the abutments, S_1 and S_2.

If vertical lines be then drawn from, S_1 and S_2, intersecting the indefinite lines, $1'\,S$ and $5'\,S$, in points, S_1 and S_2, the line, $\overline{S_1\,S_2}$, will complete the polar polygon, and any intercept, $M\,M_1$, cut off by the limits of this polygon from a vertical line, $x\,M_1$, will be of such a graphic length that

$$r.\,\overline{M\,M_1} = \text{Bending Moment at, } x.$$

Now, let us imagine the train of loads to move together a certain distance to the right. The case can be met by

supposing the train to remain stationary; whilst the bridge-span, $S_1 S_2$, is moved an equal distance to the left. The necessary modification of the figure is easily made by displacing the abutments, S_1 and S_2, into the positions, S_3 and S_4; where,

$$\overline{S_1 S_3} = \overline{S_2 S_4},$$

and then by determining as before the new closing line of the polar polygon by erecting perpendiculars, from, S_3 and S_4, to meet the indefinite lines, $1' S$ and $5'S$, in, S_3 and S_4. If the line, $\overline{S_3 S_4}$, be drawn, completing the polar polygon of moments, any intercept, $m M_1$, cut off by the limits of the new polar polygon, will determine the bending moment induced at the point, x, for which the intercept has been drawn. By the same method the influence of the train-load in any position can be fully traced out.*

The nature of the curve of moments just considered may be further analysed by taking the first differential coefficients of the series of equations of moments. [Pt. III. Ch. VII. § 4.] These differential coefficents, taken in their order, are

$$\frac{d M}{d x} = - R_2$$

$$\frac{d M}{d x} = - R_2 + P_5$$

$$\frac{d M}{d x} = - R_2 + P_5 + P_4,$$

and so on, the sixth or last differential form being

$$\frac{d M}{dx} = - R_2 + P_5 + P_4 + P_3 + P_2 + P_1.$$

The first value of, $\frac{d M}{d x}$, given above is obviously negative; hence in the interval between the lines of action of, R_2 and P_5, M, decreases as, x, increases. The differential coefficient, $\frac{d M}{d x}$, commences by being negative; but after a while in proportion as a series of positive terms, P_5, P_4, etc., are added, it will be

* This theory, in its general form, was first given by Bresse, in the *Annales des Ponts et Chaussées.*

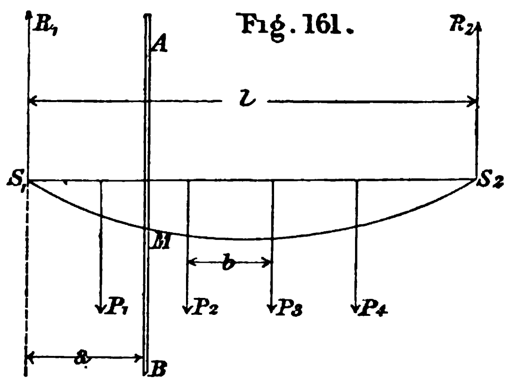
found to pass from negative to positive ; for it has been shewn that

$$R_2 < \Sigma . P.$$

After the point has been reached where, $\frac{dM}{dx}$, becomes positive, M will increase simultaneously with, x. At the origin, $M = 0$; hence the polygonal contour of moments commences at the origin, extending to a maximum distance below the line of girder, at the point, real or ideal, where, $\frac{dM}{dx}$, changes sign. It afterwards rises and joins the abutment, S_2, where the value of, M, again vanishes. This investigation supposes that the polar polygon of moments has been drawn relatively to a pole, O, so chosen that the completing line of the polygon may coincide with the centre-line of the beam.

If any of the values of, $\frac{dM}{dx}$, derived by differentiation from the different equations of moments, should happen to vanish, the contour of the polar polygon of moments must have one of its sides horizontal.

Fig. 161.



It should be remarked that since the ordinate of the curve of moments, or lower limit of the polar polygon always exists between the ends, S_1 and S_2, of the girder, the bending moment represented by it, or by its proportionate part, must be always of the same sign.

It will, moreover, be noticed that, since, $\frac{dM}{dx}$, changes sign once only, the curve of moments contains no *re-entrant* angle.

It may be required to find the law according to which the bending moment at any section, $\overline{A\,B}$, Fig. 161, distant, a, from the origin at, S_1, varies with a series of loads, P, displaced along the length of the girder, $\overline{S_1\,S_2}$. The graphic treatment of this problem has just been considered; but it may be well to look at it also from an analytical point of view. Let, R_2, represent the reaction, at S_2, which can be determined by taking moments about the point, S_1; so that, if the span be l; x_1, being the abscissa of the first load; x_2, that of the second; and so on,

$$R_2 \,.\, l = \Sigma_1 \,.\, P' x' + \Sigma_2 \,.\, P'' x'' ;$$

in which expression, $\Sigma_1 . P' x'$, represents the sum of the moments of the forces acting between the origin, S_1, and the plane of section, and, $\Sigma_2 .\, P.'' x''$, a similar sum for limits beyond that plane.

Hence,

$$R_2 = \frac{\Sigma_1 \,.\, P' x' + \Sigma_2 \,.\, P'' x''}{l} \,.$$

The bending moment at the section-plane at, a, can be found as usual by taking the moments of the forces acting between the plane and the right-hand extremity of the girder, which will be,

$$\begin{aligned} M &= R_2 [l - a] - \Sigma_2 .\, P'' [x'' - a] \\ &= \frac{l - a}{l} \,.\, [\Sigma_1 P' x' + \Sigma_2 P'' x''] - \Sigma_2 .\, P'' [x'' - a] \\ &= \frac{l - a}{l} .\, \Sigma_1 \,.\, P' x' + a .\, \Sigma_2 P'' - \frac{a}{l} \Sigma_2 P'' x'' \\ &= \frac{1}{l} \left[(l - a) \,.\, \Sigma_1 P' x' + a \, \Sigma_2 P'' (l - x'') \right]. \end{aligned}$$

Let the series of loads be displaced through a short distance, $d x' = d x''$, towards the end, S_2, of the girder. The corresponding variation in the value of, M, will be found by taking

the differential coefficient, $\frac{dM}{dx}$, remembering that the abscissa of the fixed sectional plane is constant; whilst both, x' and x'', vary by an equal amount, dx. Thus it will be seen that

$$\frac{dM}{dx} = \frac{1}{l}\left[(l-a).\Sigma_1.P' - a\,\Sigma_2.P''\right]$$

$$= \frac{a.(l-a)}{l}.\left[\frac{\Sigma_1.P'}{a} - \frac{\Sigma_2.P''}{l-a}\right].$$

Now, the quotients, $\frac{\Sigma_1.P'}{a}$ and $\frac{\Sigma_2.P''}{l-a}$, contained in this expression, represent at any given time the load per unit-length of span bearing upon the two parts into which the girder is divided by the plane of section. When, $\frac{dM}{dx}$, is positive, M, will increase with, x' and x'', that is with the general displacement of the load in the direction of, S_2. On the other hand, M, will decrease, as the general displacement of the load, x, increases, when the value of, $\frac{dM}{dx}$, is a negative quantity.

The expression for, $\frac{dM}{dx}$, remains positive, so long as

$$\frac{\Sigma_1.P'}{a} > \frac{\Sigma_2.P''}{l-a},$$

that is, so long as the load per unit-length of span measured between, S_1, and the plane of section, exceeds the value of the same unit between the limits, l and a. The maximum of M, corresponds, therefore, to the equality.

$$\frac{\Sigma_1.P'}{a} = \frac{\Sigma_2.P''}{l-a}.$$

It may happen that, while the train of loads moves forward in the direction of S_2, the load per unit-length of span on either side of the plane of section does not vary. In that case, $\frac{dM}{dx}$, is constant; or the variation in the bending moment

at the given section, due to the general displacement of the loads, obeys a uniform constant law; and the equation

$$\frac{dM}{dx} = \frac{a(l-a)}{l} \cdot \left[\frac{\Sigma_1 . P'}{a} - \frac{\Sigma_2 . P''}{l-a}\right]$$

furnishes the absolute value of the increment of M, per unit length of span moved over by the loads. Consequently, if this unit-increment in, M, be multiplied by the total displacement of the train-load, the product will give the total increment in the value of, M. The length of span traversed must be measured by the same standard as, a and l.*

Taking an example, to illustrate the above theorem, let four equal loads, P, Fig. 161, separated from each other by a constant interval, b, pass over the length of girder, $\overline{S_1 S_2}$. Let the plane of section, $\overline{A B}$, be situate at, ¼th the length of the span from the origin, S_1, so that, $a = \frac{l}{4}$.

The greatest bending moment at the section considered will correspond to the equality,

$$\frac{\Sigma_1 . P'}{a} = \frac{\Sigma_2 . P''}{l-a},$$

which gives,

$$\frac{\Sigma_1 . P'}{\Sigma_2 . P''} = \frac{1}{3},$$

and it will be seen that this relation holds at the time, when the sum of the loads on the near side of the plane is equal to ⅓ of the sum of the loads on the side of the plane removed from the origin. This is only the case when one of the loads, P, is applied between, $x = a$ and $x = o$; and the remaining three are situate between the limits, l and a.

Let, R_2, represent the reaction at S_2, due to this particular distribution of the loads. Taking moments about, S_1, and supposing the first and second loads to lie at equal distances on opposite sides of the plane of section, we find,

* So far as the author is aware, this elegant demonstration is due to M. Bresse.

$$R_2 . l = P . \left[\left(\frac{l}{4} - \frac{b}{2} \right) + \left(\frac{l}{4} + \frac{b}{2} \right) + \left(\frac{l}{4} + \frac{3}{2} . b \right) + \left(\frac{l}{4} + \frac{5}{2}\, b \right) \right] = P . [l + 4\, b] ;$$

so that,

$$R_2 = \frac{P}{l} . [l + 4\, b]$$

The greatest bending moment at the section, $\overline{A\, B}$, corresponding to the above value of, R_2, will be,

$$M = R_2 (l - a) - P \left[\frac{b}{2} + \frac{3}{2} . b + \frac{5}{2} . b \right]$$
$$= P . \left[\frac{3}{4}\ (l + 4\, b) + \frac{9}{2}\, b \right]$$

Put, $l = 30\, b$; then this equation shews that

$$M = 21 . P . b .$$

If by the preceding method the maximum values of, M, corresponding to the various sections of the girder, and positions of the train of loads have been determined, a curve of moments can be drawn of such a nature that any intercept on $\overline{A\, M}$, Fig. 161, shall graphically represent the greatest bending moment at the section through, A. This curve has also been called the envelope of moments ; since it forms a limit within which are contained the ordinates representing the bending moments at any section of the girder. The moments corresponding to the curve itself are the greatest bending moments induced at the different sections of the beam by the action of the moving train of loads. It is with a curve of this nature that we have generally to deal, when determining the greatest longitudinal stresses produced at any section, by the application of the maximum live loads to the span of a bridge-girder.

From a variety of curves of the above kind it has been deduced that the curve-envelope of moments differs very little from a parabolic curve, constructed by taking the ordinate at the centre of span as the abscissa of the vertex of the parabola,

and the span itself as the double ordinate. Let, M_0, be this central ordinate, found either by the preceding analytical process; or by the graphic processes earlier described [pp. 315—21]. Make, $M_0 = -\frac{1}{8} p l^2$; then p, will represent the load per unit-length of span, uniformly distributed over the entire length of the girder, which in the sum will adequately represent and take the place of the actual moving loads.

A French government circular, bearing the date, July, 9th, 1877, and based upon the report of a commission appointed by the, "*Conseil Général des Ponts et Chaussées*" authorises the substitution of an equivalent uniform load for the actual moving loads traversing railway bridges; and, moreover, gives a table of the values of, p, which should be used for different spans of the girders. Annexed we give this Table with the measures converted.

TABLE I.				TABLE II.			
Span, l, ft.	Value of, p, in lbs. per linear foot.	Span, l, ft.	Value of, p, in lbs. per linear foot.	Span, l, ft.	Value of, p, in lbs. per linear foot.	Span, l, ft.	Value of, p, in lbs. per linear foot.
	Single Line.		Single Line.		Two Wheels.		Four Wheels.
6·56	8052	49·21	3825	9·84	4920	9·84	4466
9·84	7045	52·50	3691	13·12	3690	13·12	3495
13·12	6844	55·70	3623	16·40	2952	16·40	2920
16·40	6576	59·06	3489	19·69	2460	19·69	2521
19-69	6374	62·34	3422	22·97	2109	22·97	2258
22·97	5972	65·62	3288	26·25	1845	26·25	2050
26·25	5569	82·02	3020	29·53	1647	29·53	1889
29·53	5236	98·43	2885	32·80	1488	32·80	1760
32·80	4898	114·83	2818	39·37	1246	39·37	1538
36·09	4630	131·24	2751	45·93	1080	45·93	1376
39·37	4362	164·05	2616	59·06	864	59·06	1153
42·65	4160	328·10	2080	82·02	657	82·02	929
45·93	3959	492·15 and over	2013	164·05	472	164·05	755

M. L'Inspecteur-Général Kleitz has further developed this method of substitution, and has calculated values of, p, which can be substituted in cases of bridges for ordinary road traffic. He supposes the bridge to be traversed by a series of heavy wagons, drawn by teams of horses. The loads due to the wagons and their contents are taken as concentrated over

the axles, and the weight of the horses themselves as concentrated at the centre of gravity of the animals. He allows each horse to weigh half a ton. The loads in the wagons are divided into two classes ;—in the first, a two-wheeled wagon containing 11 tons of material is supposed to be drawn by a team of five horses, yoked in tandem fashion, the points of concentrated animal weight being separated by three mètres ; —in the other case, a four-wheeled wagon with axles three mètres apart, each loaded with eight tons weight, is drawn by a team of eight horses, yoked two abreast. By a consideration of the greatest bending moments, produced by loads of this kind, he arrives at the corresponding values of, M_o, the central ordinate of the curve-envelope of moments ; and subsequently, by aid of the equation,

$$M_o = -\frac{p\,l^2}{8},$$

he obtains the values of, p, related to spans of different length, which are embodied in the Table, II., annexed, p. 327.

5. SOLID GIRDERS FIXED AT BOTH ENDS.—If the ends of a girder are not only supported, but at the same time firmly fixed and held down at the abutments, S_1, S_2, Fig. 162, the forces acting at S_1 and S_2, may be looked upon as equivalent to two simple reactions and two attendant couples, all unknown. The moments due to '*encastrement*' will coexist with the reactions due to the loads.

Let us suppose a girder, such as that shewn in Fig. 162, subjected to a load, P, concentrated at the centre of span and an uniform load, p, per unit of length of the girder. The beam being allowed to be in equilibrium, the forces acting upon it must, when added together, balance each other ; so that

$$R_1 + R_2 = P + pl,$$

and since, by reason of the symmetrical distribution of loads, $R_1 = R_2$; each of the reactions will be equal to,

$$\frac{1}{2}\Big[P + pl\Big] = \frac{P}{2} + pa.$$

Let, M, be the bending moment induced at any section distant, x, from the centre of span, chosen as the origin of co-ordinates ;—then

$$M = -\frac{P}{2}[a-x] - p\,a\,[a-x] + p\,(a-x)\,\tfrac{1}{2}\,[a-x] + \mu_2,$$

in which expression the moment, μ_2, attendant on the reaction, R_2, will be right-handed as shewn in the figure, and therefore

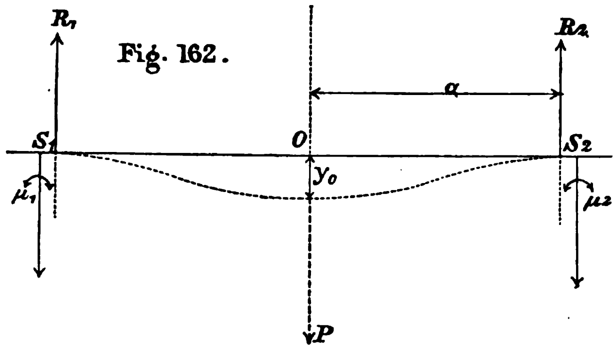


Fig. 162.

opposite in sign to the couple expressed by the first term, $-\frac{P}{2}[a-x]$. The above value of, M, being reduced, takes the form,

$$er^2\frac{d^2y}{dx^2} = M = \frac{-P}{2}[a-x] - \frac{p}{2}[a^2 - x^2] + \mu_2$$

By a first integration,

$$er^2 . \frac{dy}{dx} = \frac{-P}{2} . \left[ax - \frac{x^2}{2}\right] - \frac{p}{2} . \left[a^2 x - \frac{x^3}{3} + \mu_2\, x\right]$$

there being no constant, since, when $x = o$, $\frac{dy}{dx} = o$. But owing to the fact that the beam is horizontally fixed at the points of support, $\frac{dy}{dx} = o$, when $x = a$; hence,

$$\mu_2\, a - \frac{P}{2} \cdot \frac{a^2}{2} - \frac{pa^3}{3} = o\,;$$

or,

$$\mu_2 = \frac{Pa}{4} + \frac{pa^2}{3}.$$

Substituting this value of, μ_2, in the general value of $\frac{dy}{dx}$, we find

$$er^2.\frac{dy}{dx} = \frac{-P}{4}.[ax - x^2] - \frac{p}{6}.[a^2x - x^3].$$

Integrating a second time and determining the constant added, by the relation, $y = o$, when $x = a$,

$$er^2y = \frac{P}{4}.\left[\frac{a^3}{6} + \frac{x^3}{3} - \frac{ax^2}{2}\right] + \frac{p}{6}\left[\frac{a^4}{4} + \frac{x^4}{4} - \frac{a^2x^2}{2}\right].$$

In this equation put $x = o$; then, the value of the ordinate of deflection at the centre of span will be found to be,

$$y_o = \frac{a^3}{24\,er^2}[P + p\,a].$$

If, $p = o$, which is equivalent to the suppression of the uniform load, the central deflection due to concentrated load, P, separately taken, is,

$$y_o = \frac{a^3}{24\,er^2}.P;$$

the value of the same ordinate, when the ends of the girder are simply supported and not fixed, is [Pt. III. Ch. VII. § 1],

$$y_o = \frac{a^3}{6\,er^2}P,$$

or four times its value in the case under consideration.

If, $P = O$, which is equivalent to the suppression of the fixed load, the central deflection due to uniform load, separately taken, will be

$$y_o = \frac{pa^4}{24\,er^2};$$

the value of, y_o, when the ends are not fixed and the load uniform, is [Pt. III. Ch. VII. § 1],

$$y_o = \frac{5}{24}\frac{pa^4}{er^2},$$

or five times its value under the same conditions, after the ends have been firmly fixed down.

In order that the central deflections, produced by the concentrated load, P, and the uniform load, p, separately taken, may be equal; it is necessary that,

$$\frac{P.a^3}{24.er^2} = \frac{p\,a^4}{24\,er^2}$$

that is,

$$P = pa = \frac{pl}{2};$$

or the concentrated load, P, should be equal to half the distributed load, pl.

Taking the general equation of moments,

$$M = -\frac{P}{2}[a - x] - \tfrac{1}{2}p\,.\,[a^2 - x^2] + \mu_2;$$

which, on substituting,

$$\mu_2 = \frac{Pa}{4} + \frac{pa^2}{3},$$

becomes,

$$M = \frac{-P}{4}[a - 2x] - \frac{p}{6}[a^2 - 3x^2], \qquad (1)$$

it will be seen that the value of, M, varies with the abscissa, x, of the plane of section.

When, $x = o$, M, is evidently negative; and as x, increases in value, M, approaches zero, to which it is ultimately equal at a point, x, determined by the equation,

$$\frac{P}{4}[a - 2x] + \frac{1}{6}p\,.\,[a^2 - 3x^2] = o \qquad (2)$$

This quadratic in, x, can easily be solved and the point found for which, $M=O$, and consequently where, $\frac{d^2y}{dx^2}=o$. It will coincide with a point of inflexion; since, between the limits, x and o, for values of x determined by the last equation $\frac{d^2y}{dx^2}=\frac{M}{EI}$, is negative; whereas beyond, x, the value of $\frac{d^2y}{dx^2}$, remains positive.

The curve of deflection will be concave towards the axis of, x, for the part of the girder, relatively to which, $\frac{d^2y}{dx^2}$, is a negative quantity, and convex towards the same axis for the rest of the curve.

An approximation to the position of the point of inflexion corresponding to the equality, $M=er.^2\frac{d^2y}{dx^2}=O$, can be made by considering that if, P, act alone and, p, be suppressed; the point of inflexion must lie nearer the centre of the span, at which the entire load is concentrated, than when both loads act together. Again, when only uniformly distributed load is applied to the girder, the point of inflexion will be situate nearer the support, S_2. When however both kinds of load are simultaneously brought to bear upon it, the locus of zero-moment and inflexion of curvature will occupy a mean between its positions corresponding to concentrated and uniformly distributed loads.

Now, when $p=o$, that is when, P, acts alone, equation [2] furnishes the relation,

$$\frac{P}{4}\cdot[a-2x]=O,$$

from which, we find, $x=0{\cdot}5\,a$.

When, $P=O$, we have

$$\frac{p}{6}[a^2-3x^2]=o,$$

whence, $x=0{\cdot}577a$. Therefore, when both, P and p, act

simultaneously upon the girder, the abscissa of the point of inflexion will be situate somewhere between the limits, $x = 0{\cdot}5a$ and $x = 0{\cdot}577a$.

The maxima of, M, correspond to the abscissæ, $x = o$, and $x = a$, and are respectively equal to,

$$M_o = -\left[\frac{P}{4}a + \frac{pa^2}{6}\right]$$

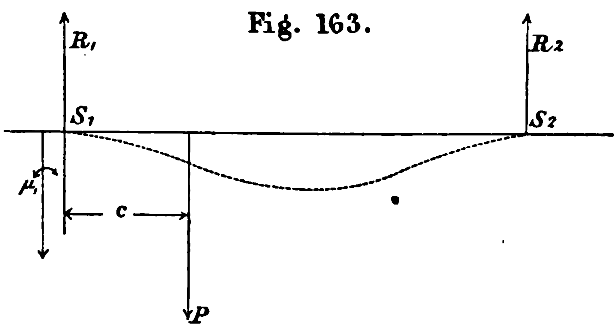
and

$$M_a = \left[\frac{Pa}{4} + \frac{pa^2}{3}\right].$$

When, $p = o$, these values of, M, are equal in magnitude but opposite in sign; whereas, when, p, exists, the greatest bending moment is found in the vicinity of the support-abutments, or *encastrements.*

6. SOLID GIRDERS FIXED AT ONE END.—As a second example, let there be given a beam supported at one end, S_2, and fixed horizontally at the other, S_1. Let a load, P, be applied anywhere along the beam at a section, distant, c, from the abutment, S_1, taken as origin of co-ordinates, [Fig. 163].

Fig. 163.



Since the beam is supposed to be in equilibrium under the action of the forces applied to it, the vertical projections of these forces must in the sum vanish; hence

$$R_1 + R_2 - P = 0.$$

Again, if moments be taken about the origin, S_1, of all the

forces acting on the girder, the sum of these moments must vanish; so that

$$R_2 . 2a - Pc + \mu_1 = 0$$

It will be seen (Fig. 163) that the sense of the attendant couple, μ_1, is the same as that of the moment arising from the reaction, R_2; and, therefore, it takes the same sign.

If the section of the beam be invariable and the coefficient of elasticity constant, the differential equation of the curve, assumed by the deflected neutral fibre, will be given by the usual form,

$$E\,I\,\frac{d^2y}{dx^2} = M$$

Now let, x, be the abscissa of any plane of section between the limits, $2a$ and o, of the beam, and let moments be taken relatively to this section. The value of the resultant moment will depend upon the position of the plane of section chosen; for, taking the moments of all forces applied on the right of the plane at, x; it is manifest that, whilst, R_2, must always act on the right of the sectional plane, wherever that may be pitched, the force, P, will only act on the right of those planes of section, which are situated between the limits, $x=c$ and $x=o$. The general form,

$$M = R_2\,[2a - x] - P\,[c - x],$$

will, however, apply to all possible positions of the plane of section; provided it be well understood that the second term of the right-hand member of the above equation, namely $P. [c-x]$, vanishes for all planes situate between the limits, $x=2a$, and $x=c$.

By a first integration of the general equation,

$$M = E\,I\,\frac{d^2y}{dx^2} = R_2 . [2a - x] - P . [c - x],$$

we shall find,

$$E\,I\,\frac{dy}{dx} = R_2 \int_o^x (2a - x)\,dx - P . \int_o^x (c - x)\,dx\,;$$

no constant being required, since when, $x = o$, $\frac{dy}{dx} = o$.

Now, in regard to the first of these integrable forms, it is clear that, no matter where the plane of section may be taken, the part-moment, $R_2 (2a - x)$, due to the reaction, R_2, always obtains; therefore, the factor $(2a - x)$, may be considered to preserve the same functional form for all values of, x. Generally, therefore,

$$R_2 . \int_o^x (2a - x)\, dx = R_2 \left[2\, a\, x - \frac{x^2}{2} \right].$$

The second integrable form; namely,

$$P. \int_o^x (c - x)\, dx,$$

exists only on the right of such planes of section as lie between the limits, $x = c$, and $x = o$. Hence, a distinction must be made. If, $x < c$,

$$P. \int_o^x (c - x)\, dx = P. \left[c\, x - \frac{x^2}{2} \right]$$

If, however, $x > c$, $P \int_o^x (c - x)\, dx$, must be integrated according to two functional forms. First, as x, increases from, $x = o$, to $x = c$, the usual form exists, leading, when integrated, to

$$P. \int_o^x (c - x)\, dx = P. \left[(c\, x - \frac{x^2}{2} \right].$$

But, beyond the point, for which, $x = c$, the factor, P, vanishes; so that when $x > c$, the integration must be made by the addition of the two parts,

$$P. \int_o^x (c - x)\, dx = P. \int_o^c (c - x)\, dx + 0. \int_c^x (c - x)\, dx. = \frac{P . c^2}{2}$$

Hence, if $V = P \cdot \int_0^x (c - x)\, dx$,

$$E\,I \frac{dy}{dx} = R_2 \cdot \int_0^x [2a - x]\, dx - P \int_0^x (c - x)\, dx$$

$$= R_2 \cdot \left[2ax - \frac{x^2}{2}\right] - V;$$

where, V, is a symbol representing two functional forms; namely,

$$P \cdot \left[cx - \frac{x^2}{2}\right];\text{—for values of, } x, \leqq c$$

and,

$$\frac{P \cdot c^2}{2}; \text{ — for values of, } x, \geqq c.$$

By a second integration of the above general form,

$$E\,I \cdot y = R_2 \cdot \int_0^x \left[2ax - \frac{x^2}{2}\right] \cdot dx - \int_0^x V \cdot dx$$

$$= R_2 \cdot \left[a\,x^2 - \frac{x^3}{6}\right] - \int_0^x V \cdot dx,$$

in which, V, admits of two different interpretations for different limits of, x, as graphically shewn in Fig. 164, from which it will be seen that, as already proved, V, becomes constant beyond the limit, $x = c$, wherefore,

$$\int_0^x V \cdot dx = P \cdot \int_0^c \left[cx - \frac{x^2}{2}\right] \cdot dx + \frac{P}{2} \int_c^x c^2 \cdot dx$$

$$= \frac{P \cdot c^2}{2} \cdot \left[x - \frac{c}{3}\right].$$

Hence, there will be two equations to the curve of flexure corresponding to different parts of the girder. When, $x \leqq c$,

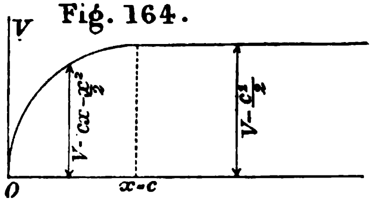
$$E\,I \cdot y = R_2 \cdot x^2 \left[a - \frac{x}{6}\right] - P \cdot x^2 \cdot \left[\frac{c}{2} - \frac{x}{6}\right],$$

and, when, $x \geqq c$,

$$E I . y = R_2 . x^2 . \left[a - \frac{x}{6}\right] - \frac{P . c^2 .}{2} \left[x - \frac{c}{3}\right].$$

In both these equations, R_2, is a quantity not yet determined. If, therefore, in either equation some particular value of, y, corresponding to a particular value of, x, be substituted, the reaction, R_2, will be thereby fixed in amount. The first

Fig. 164.



equation defines nothing about R_2; for, when, $x = o$, $y = o$, and the equation becomes an identity. But by the second equation, when, $x = 2a$, $y = o$; so that

$$R_2 . 4 a^2 \left[a - \frac{2a}{6}\right] = \frac{P . c^2}{2} . \left[2a - \frac{c}{3}\right]$$

or,

$$R_2 . \frac{8}{3} . a^3 = \frac{P . c^2}{2} \left[2a - \frac{c}{3}\right].$$

Let, $c = r . a$; then

$$R_2 . \frac{8}{3} a^3 = \frac{P . r^2 a^2}{2} . \left[2a - \frac{ra}{3}\right]$$

$$= \frac{P . r^2 a^3}{6} . [6 - r],$$

whence,

$$R_2 = \frac{P . r^2}{16} . [6 - r],$$

and by a former relation,

$$R_1 = P - R_2 ; \; \mu_1 = P . c - R_2 . 2a.$$

Uniform Load, one end of girder fixed :—

If instead of a concentrated load, P, a uniform load, p, were applied to each unit-length of the girder, the following

relations could be deduced by taking the vertical projections of the loads and reactions and equating their sum to zero, and, further, taking the moments of the same forces relatively to the origin at, S_1, and stating the condition that the sum of all the moments acting at the section through, S_1, must vanish. Thus, we have

$$R_1 + R_2 - 2\,p\,a = 0$$

and,

$$R_2 . 2\,a - 2\,p\,a . a + \mu_1 = 0\,;$$

hence,

$$R_2 = \frac{2\,p\,a^2 - \mu_1}{2\,a}.$$

Taking the usual equation of moments,

$$E\,I\,\frac{d^2 y}{dx^2} = R_2\,(2\,a - x) - p\,(2\,a - x).\left[\frac{2\,a - x)}{2}\right]$$

$$= \left(\frac{2\,p.\,a^2 - \mu_1}{2\,a}\right)[2\,a - x] - \frac{p}{2}\,[2\,a - x]^2$$

By a first integration,

$$E\,I\,\frac{dy}{dx} = \left(\frac{2\,p\,a^2 - \mu_1}{2\,a}\right)\left[\,2\,a\,x - \frac{x^2}{2}\right] - \frac{p}{2}\left[4\,a^2 x + \frac{x^3}{3} - \frac{4\,a\,x^2}{2}\right]$$

and, integrating again,

$$E\,I.\,y = \left(\frac{2\,p\,a^2 - \mu_1}{2\,a}\right)\left[a\,x^2 - \frac{x^3}{6}\right] - \frac{p}{2}\left[2\,a^2 x^2 + \frac{x^4}{12} - \frac{2}{3}\,a\,x^3\right].$$

Now, when $x = 2\,a$; $y = o$, wherefore,

$$\left(\frac{2\,p\,a^2 - \mu_1}{2\,a}\right)\left[4\,a^3 - \frac{8\,a^3}{6}\right] = \frac{p}{2}\left[8\,a^4 + \frac{16\,a^4}{12} - \frac{16\,a^4}{3}\right],$$

or,

$$(2\,p\,a^2 - \mu_1)\left[4 - \frac{4}{3}\right]a^2 = 4\,p\,a^4\,;$$

that is,

$$\frac{16}{3} p a^4 - \frac{8}{3} a^2 . \mu_1 = 4 p a^4 ;$$

which gives,

$$\mu_1 = \frac{p a^2}{2}.$$

This value of, μ_1, being positive, agrees in kind with the moment due to the reaction, R_2. By the given equations of equilibrium for this case,

$$R_2 = \frac{2 p a^2 - \mu_1}{2 a} = \frac{2 p a^2 - \frac{1}{2} p a^2}{2 a} = \frac{3}{4} . p a = \frac{3}{8} p l.$$

$$R_1 = 2 p a - R_2 = \frac{5}{4} p a = \frac{5}{8} . p l.$$

Points of Inflexion and Maximum Ordinates of Deflection :— It has been shewn [p. 336—37] that there are in general two equations to the curve of flexure of the neutral fibre of the girder, which correspond respectively to the two parts into which the length of the span is divided by the plane of section at, x. This double form of the equation to the deflected curve exists in virtue of the discontinuity of the functional forms, assumed by the expression of moments relatively to different parts of the beam. Substituting in the first of these two equations, which obtains for the limits, $x = c$, and $x = o$, the value of the reaction, R_2, as previously determined ; namely, in the case of concentrated loads,

$$R_2 = \frac{P}{16} . r^2 . (6 - r),$$

and putting as before,

$$c = r a,$$

we find

$$E I . y = \frac{P}{2} \frac{(2 - r)}{48} . [x^3 (8 + 4 r - r^2) - 6 r a x^2 (4 - r)].$$

In like manner, by similar substitutions in the second equation to the curve of flexure, corresponding to the limits, $x = 2 a$, and $x = c$, there results the form,

$$E I . y = \frac{P}{2} . \frac{r^2}{48} . [x^3 (r - 6) + 6 (6 - r) a x^2 - 48 a^2 x + 16 r a^3]$$

In order to find where the curve of flexure becomes horizontal, it is necessary to examine where the values of, $\frac{dy}{dx}$, derived from these equations, vanish.

Differentiating, therefore, the first equation furnishes the relation,

$$E I \frac{dy}{dx} = \frac{P}{2} \cdot \frac{(2 - r)}{16} [x^2 (8 + 4 r - r^2) - 4 r a x (4 - r)];$$

whilst the second gives,

$$E I \frac{dy}{dx} = \frac{P}{2} \cdot \frac{r^2}{16} \cdot [x^2 (r - 6) + 4 (6 - r) a x - 16 a^2].$$

Hence, between the limits, c and o, the coefficient, $\frac{dy}{dx}$, vanishes for the values,

$$x = o, \text{ and, } x = \frac{4 r a (4 - r)}{8 + 4 r - r^2}.$$

The latter value of, x, must necessarily be less than, $r a$, in order that the form, from which it has been derived, may obtain. Wherefore, this value of, x, is subject to the condition

$$\frac{4 r a (4 - r)}{8 + 4 r - r^2} < r a.$$

The denominator is always positive for values of r, comprised between, o and 2; and consequently the inequality may be put in the form,

$$16 - 4 r < 8 + 4 r - r^2$$

or,

$$8 - 8 r + r^2 < o,$$

that is,

$$(4 - r)^2 - 8 < o,$$

from which is deduced

$$(4 - r) < \sqrt{8}$$

and

$$r > 4 - 2 \sqrt{2}$$

or

$$r > 1 \cdot 172.$$

It may be, therefore, inferred that the maximum ordinate, y, occurs between the point of application of the load and the origin, when the load is applied at a point removed from the origin by a distance not less than, $c = 1 \cdot 172\, a$; and its absolute value can be then found by substituting,

$$x = \frac{4\, r\, a\, (4 - r)}{8 + 4\, r - r^2},$$

in the first form of the equation to the curve of flexure.

When, however, $r < 1 \cdot 172$, the maximum ordinate of deflection will be found in the part of the curve defined by the second of the two equational forms, which, by equating, $\frac{dy,}{dx}$ to zero, furnishes the relation,

$$x^2 (r - 6) + 4 (6 - r)\, a\, x - 16\, a^2 = o\,;$$

or,

$$x^2 - 4\, a + \frac{16\, a^2}{6 - r} = o,$$

from which it will be found that,

$$x = 2\, a \left[1 \pm \sqrt{\frac{2 - r}{6 - r}}\right],$$

where the *minus* sign is the only one admissible.

In the particular case, when the load is applied at the centre of span; that is, when, $r = 1$, the above value of, x, becomes,

$$x = 2\, a\, [1 - \sqrt{\tfrac{1}{5}}] = 1 \cdot 106 . a.$$

The absolute value of the maximum ordinate of deflection corresponding to this particular value of, x, can be determined by substituting,

$$x = 1 \cdot 106\, a,$$

in the second form of the general equation to the curve of flexure. The result will be found to be,

$$y_o = \frac{P . a^3}{6 \sqrt{5} . e\, r^2},$$

where as usual $e\,r^2 = E\,I$, and, y_o, is seen to bear a value intermediate between those previously found [Pt. III. Ch. VII. §§ 1, 5], in the cases of beams respectively supported and fixed at both ends.

The same problem may be solved in respect to those cases where the loads are uniformly distributed; when, as already demonstrated [pp. 338—39].

$$E\,I\frac{d^2y}{dx^2} = R_2(2\,a - x) - \frac{p}{2}\,(2\,a - x)^2.$$

$$= \frac{3}{4}.\,p\,a.\,(2\,a - x) - \frac{p}{2}.(2\,a - x)^2$$

$$= \frac{p}{4}\,[5\,a\,x - 2\,x^2 - 2\,a^2].$$

By a first integration,

$$E\,I\frac{dy}{dx} = \frac{p}{4}\left[\frac{5\,a\,x^2}{2} - \frac{2}{3}.\,x^3 - 2\,a^2.\,x\right].$$

To find where the tangent to the curve of deflection becomes horizontal, we must make, $\frac{dy}{dx} = o$, which leads to the relation,

$$\frac{5\,a\,x^2}{2} - \frac{2}{3}\,x^3 - 2\,a^2.\,x = o,$$

or,

$$x^2 - \frac{15}{4}.\,a\,x + 3\,a^2 = o,$$

from which is deduced,

$$x = a\left[\frac{15 \pm \sqrt{33}}{8}\right],$$

and taking the *minus* as the only admissible sign,

$$x = 1{\cdot}156\,a.$$

The maximum ordinate corresponding to this value of, x, can be found by means of a second integration and subsequent substitution.

Thus,

$$\int E\,I.\frac{dy}{dx} = E\,I.y = \frac{p}{4}.\left[\frac{5\,a\,x^3}{6} - \frac{2\,x^4}{12} - \frac{2\,a^2 x^2}{2}\right]$$

$$= \frac{p.\,x^2}{24}\,[5\,a\,x - x^2 - 6\,a^2].$$

Now, the term,

$$-\,[x^2 - 5\,a\,x + 6\,a^2],$$

is resolvable into factors ; for, if, z_1 and z_2, be the roots of this quadratic, we have,

$$z_1 + z_2 = 5\,a$$
$$z_1\,z_2 = 6\,a^2,$$

which equations are satisfied by the values,

$$z_1 = 2\,a, \text{ and, } z_2 = 3\,a.$$

Hence,

$$E\,I.y = \frac{-\,p\,x^2}{24}\,[\,(x - 2\,a)\,(x - 3\,a)].$$

When, $x = 1{\cdot}156\,a$, the value of, y, furnished by this equation becomes,

$$y_o = 0{\cdot}0866\,\frac{p\,a^4}{e\,r^2},$$

which is a value of the maximum ordinate of deflection intermediate between the values of, y_o, previously found [Pt. III. Ch. VII. §§ 1, 5], for the cases of beams respectively supported and fixed at both ends.

Bending Moments and Shearing Forces:—For a fixed load, P, concentrated at a section, distant, c, from the end, S_1, of the girder, the bending moment induced at, S_1, is [p. 339],

$$-\,\mu_1 = R_2\,.\,2\,a - P.\,c,$$

or, substituting the values,

$$R_2 = \frac{P}{16}.\,r^2.\,(6 - r)\,;\; c = r\,a,$$

$$-\,\mu_1 = \frac{-\,P}{8}\,r\,a\,[(r - 2).\,(r - 4)].$$

The moment at a point immediately on the right of the sectional plane at, c, will be,

$$M_c = R_2(2a - c)$$
$$= \frac{P.r^2.a}{16}[(6 - r)(2 - r)].$$

The moment at the supported end, S_2, is *nil.*

In the case of uniform load the bending moment at any section, x, is given by the expression,

$$M_x = R_2(2a - x) - p(2a - x)\frac{(2a - x)}{2};$$

or, since in this case,

$$R_2 = \frac{3}{4}.pa$$

$$M_x = \frac{p}{8}[-(2x)^2 + 5a.(2x) - 4a^2]$$

$$= -\frac{p}{8}[(2x - 4a)(2x - a).$$

By the form of this equation it will be seen that, $M_x = O$, when $x = \frac{a}{2}$, and again when $x = 2a$;—moreover, M_x, is a maximum when $x = o$; or when, $x = \frac{5a}{4}$. Making these substitutions in the expression for, M_x, we find the corresponding maxima to be, when, $x = o$,

$$M_o = \frac{-p.a^2}{2},$$

and when $x = \frac{5}{4}a$,

$$M_x = \frac{9}{32}.pa^2.$$

Further, the general form of expression for, M_x, shews that, M_x, and consequently, $\frac{d^2y}{dx^2}$, change sign at the section determined by the abscissa,

$$x = \frac{a}{2} = \frac{l}{4}.$$

There is, therefore, a point of inflexion on the curve of deflection at $\frac{1}{4}$th the span from the origin at, S_1. Up to the limit, $x = \frac{l}{4}$, the curve turns its convexity towards the axis of, x; beyond this point it turns its concavity towards the same line.

If, F, represent the shearing force for a concentrated load, P, at, c; and for any section between the limits, c and o,

$$F' = R'_2 - P.$$

For the same section and a uniform load,

$$F'' = R''_2 - p\,(2\,a - x).$$

When both loads act together under the same conditions,

$$F = F' + F'' = R'_2 + R''_2 - [P + p\,(2\,a - x)],$$

where, R'_2 and R''_2, are the particular values of, R_2, corresponding respectively to concentrated and uniform load. The term, P, will disappear from the value of, F, for sections comprised between the limits, $x = 2\,a$, and $x = c$.

7. VERIFICATION OF THE STABILITY OF STRUCTURES.—Earlier in this work it has been shewn that the longitudinal stress produced in a beam in virtue of the existence of bending moments can be expressed in the form, [p. 249].

$$\frac{t}{\omega} = \frac{M\,.\,y}{I},$$

where, y, is the perpendicular distance of the elemental fibre considered, above or below the axis of flexure. If the form of cross-section be constant in area and symmetrical in form with respect to the axis of flexure, the unit-stress attains a maximum at the section of greatest bending moment, because in that case, I and y, are constant, and, M, alone varies from section to section. Let this maximum be represented by the

symbol, M_o; then, if, h, be the depth of the beam and, $y = \frac{h}{2}$, the maximum unit-stress will be expressed by,

$$\frac{t}{\omega} = \frac{M_o}{I} \cdot \frac{h}{2}.$$

Suppose the greatest safe-working tension of the material employed to be represented by f_t, whilst its greatest unit of compressive resistance under working conditions is f_c; and let, f, express the lesser of these two units of stress. It will be necessary, in order that the beam may be insured against fracture, that

$$\frac{M_o}{I} \cdot \frac{h}{2} \leqq f.$$

In this way provision is made against the beam giving way by reason of the bending moments acting at its various sections. Moreover, there will be a surplus of strength at sections where the bending moments developed are less than, M_o; or again where the unit of resistance is greater than, f.

But, notwithstanding that the beam may be made sufficiently strong to resist the action of bending moments, cases may arise where the section given may be of insufficient strength to withstand the incisive action of the shearing force.

The method of computing the shearing stress per unit-area of cross-section has been previously explained, where it was proved that, generally speaking, this stress is *nil* at the upper and lower limits of cross-section, increasing in effect on elements situate nearer the axis of flexure, at which it reaches a maximum. This maximum in a rectangular form of section was shewn to be [p. 269],

$$\frac{s}{\omega} = \frac{3}{2} \cdot \frac{F}{\Omega},$$

in which expression, F, symbolises the shearing force, and, Ω, the area of cross-section. If, therefore, Ω, be constant, the maximum shearing stress will take place at the section of

greatest shearing force. Let the maximum shearing force be represented by, F_o, and let, f_s, be the safe-working stress of the material per unit area of cross-section; then it is necessary that,

$$\frac{3 \cdot F}{2 \Omega} \leqq f_s.$$

When the form of cross-section assumes the shape of a double, *T*, and is, moreover, symmetrical with respect to the axis of flexure, it has been shewn [p. 271] that the shearing force is distributed over the section in such a way that its effect on the flanges may be considered *nil*, and its distribution over the web uniform; so that in this case, Ω, being the area of the web, and, F_o, the greatest shearing force, it is necessary that,

$$\frac{F_o}{\Omega} \leqq f_s.$$

Generally speaking a section sufficiently strong to resist the maximum bending moment, has ample strength to withstand the action of shearing forces.

When, however, the section is constant but unsymmetrical in form above and below the axis of flexure, let, y_1, be the maximum ordinate of the upper limit of any cross-section, and, y_2, that of the lower limit, both referred to the axis of flexure. Let, M_1, be the maximum positive moment; that is of the kind which produces tension in the lower and compression in the upper fibres of the beam. Let, M_2, be the greatest negative moment. The maximum, M_1, will produce a maximum unit of tension proportional to, M_1 and y_2, expressed by the formula,

$$\frac{t_1}{\omega} = \frac{M_1 y_2}{I}.$$

The same moment will also give rise to a maximum unit-compression equal to

$$\frac{p_1}{\omega} = \frac{M_1 y_1}{I}.$$

Similarly, the maximum units of tension and compression due to the negative moment, will be expressed by,

$$\frac{t_2}{\omega} = \frac{M_2 y_1}{I}; \quad \frac{p_2}{\omega} = \frac{M_2 y_2}{I}.$$

Let, $\frac{t}{\omega}$, represent the greater of the two values, $\frac{t_1}{\omega}$ and $\frac{t_2}{\omega}$, and, $\frac{p}{\omega}$, that of the two, $\frac{p_1}{\omega}$ and $\frac{p_2}{\omega}$; then, if, f_t be the safe-working tension, and, f_p, the safe-working compression, the two conditions of strength will take the forms,

$$\frac{t}{\omega} \leqq f_t, \text{ and, } \frac{p}{\omega} \leqq f_p.$$

The locus of the axis of flexure in any section is determined by the fact that it traverses the centre of gravity of the materialised area of the section; and is likewise normal to the longitudinal section of the beam.

If the cross-section of the beam, Ω, remain constant, it is manifest that the stresses due to shearing force will not be affected by a change in the form or symmetry of the section.

When the area, as well as the symmetry of the section varies, a case already considered under the head of *Beams of Uniform Strength*, the stresses expressed as before by

$$\frac{t}{\omega} = \frac{My}{I}; \quad \frac{s}{\omega} = \frac{3}{2} \cdot \frac{F}{\Omega}; \quad \frac{s}{\omega} = \frac{F}{\Omega},$$

will all vary from one section to another, and, therefore, it would be necessary, in order to ensure the stability of the beam, to investigate the relations,

$$\frac{t}{\omega} \leqq f_t; \quad \frac{s}{\omega} \leqq f_s; \quad \frac{p}{\omega} \leqq f_p,$$

for each section of the beam in particular, being careful to take the greatest value of the ordinate, y, in conjunction with the local bending moment.

The two conditions of stability,

$$\frac{My}{I} \leqq f \text{ and, } \frac{F}{\Omega} \text{ or } \frac{3}{2}\,\frac{F}{\Omega} \leqq f_s,$$

do not connect the dimensions of the cross-section, as implied in the symbols, so intimately that, when converted into equalities, they may enable us to determine the depth and breadth of section required. An example will prove this statement very conclusively. Let the section of the beam be rectangular in form, and constant in area, of a breadth, b, and depth, h. In this case, $I = \frac{b\,h^3}{12}$. Make, $M = 50$ tons; $F = 10$ tons, and let the unit-working-stress, to which the material of the beam can be safely subjected, be equal to, 4 tons per square inch.

Here we have,

$$\frac{M}{I} \cdot \frac{h}{2} = f,$$

or,

$$\frac{50}{\frac{bh^3}{12}} \cdot \frac{h}{2} = 4,$$

whence,

$$b\,h^2 = 75. \qquad (1)$$

Again,

$$\frac{3}{2} \cdot \frac{F}{\Omega} = 4,$$

gives

$$\frac{3\,.\,10}{2\,.\,b\,.\,h} = 4,$$

whence,

$$b.h = 3 \cdot 75. \qquad (2)$$

Now, dividing (1) by (2),

$$\frac{b\,h^2}{b\,h} = h = \frac{75}{3{\cdot}75} = 20 \text{ inches},$$

and therefore,

$$b = \frac{3 \cdot 75}{20} = 0 \cdot 187 \text{ inch}.$$

These two values of, h and b, found by combining the two equations of stability, are impracticable. Hence, it is necessary to proceed on the supposition that the two equations are not

consistently independent; and thereupon to establish some *à priori* relation between, b and h. Let, $h = 10.b$. Then, equation, (1) gives,

$$b h^2 = \frac{h^3}{10} = 75,$$

and

$$h = \sqrt[3]{750} = 9'', \text{ very approximately}$$

$$b = \frac{h}{10} = \left(\frac{9}{10}\right)'' \quad \text{"}$$

It then becomes necessary to examine whether the relation,

$$\frac{3}{2} \cdot \frac{F}{b h} \leqq 4,$$

is satisfied by these particular values of, b and h;—which can be immediately ascertained by making the requisite substitution.

In using the equations of stability of a beam, it sometimes happens that certain of the external forces are unknown, and consequently the values of, M and F, which are involved in these equations, cannot be determined by an *à priori* calculation. Such is the case when the weight of the beam itself must be included in the load; for, the weight of a beam depends upon its section, which at first is an unknown quantity.

There are three ways in which this defect in the calculation may be made good:—

1°. By calculating the bending moments, and shearing forces, making abstraction of the weight and influence of the beam; and afterwards repeating the same calculations with the weight of beam added as deduced from the dimensions given by the first calculation. A third calculation, or even a fourth and fifth may then be made, taking in each case the weight of beam deduced from the dimensions found by the preceding calculation. Two consecutive calculations will at length give dimensions for the section so nearly equal as to permit us to adopt either for the definite proportions of the beam, and the weight deduced from them as its definite weight.

2°. Secondly, an ideal weight, fixed after similar types of beam, may be substituted for the actual weight, which method is generally sufficiently accurate for all practical purposes.

3°. The weight may be expressed as a function of the unknown area of cross-section, and introduced under this form into the general equations of stability. For example, let p_o, represent the weight per unit of length and area of the beam; or in other words per unit of volume. If the total length of beam be, $2a$, its weight may evidently be considered equivalent to a uniformly distributed load, $p_o \Omega$, per unit of length, where Ω, symbolises the area of cross-section. If, besides, there be a permanent load, p, per unit-length, the maximum bending moment due to this double uniform load will be,

$$M_o = \tfrac{1}{2} . p a^2 + \tfrac{1}{2} . p_o \Omega . a^2,$$

and the greatest shearing force,

$$F_o = p a + p_o \Omega . a,$$

wherefore, the equations of stability become,

$$\frac{M_o}{I} . \frac{h}{2} = \frac{h}{4I} [p a^2 + p_o \Omega a^2] \leqq f$$

$$\frac{3}{2} . \frac{F_o}{\Omega} = \frac{3}{2} \left[\frac{p a}{\Omega} + p_o a \right] \leqq f_s.$$

Suppose the square of radius of gyration of the section be, r^2;—and let, m, be a factor depending on the form of the cross-section, such that,

$$m = \frac{r^2}{\left(\frac{h}{2}\right)^2} = \frac{I}{\Omega} \div \left(\frac{h}{2}\right)^2$$

and, therefore,

$$I = m . \frac{h^2}{4} . \Omega.$$

Substituting this value of, I, in the first equation of stability, we find,

$$\frac{1}{m . h \Omega} [p a^2 + p_o \Omega . a^2] \leqq f,$$

or,

$$\frac{p}{\Omega} + p_o \leqq \frac{m\,h}{a^2} . f,$$

and, by the second equation of stability,

$$\frac{p}{\Omega} + p_o \leqq \frac{2}{3\,a} . f_s.$$

It must be well observed that, m, is a function of the transverse section of the beam, having the form, already defined,

$$m = \frac{I}{\Omega} \div \left(\frac{h}{2}\right)^2.$$

If, therefore, we suppose the breadth of the section, b, to bear a given ratio to the depth, h; m, becomes at once expressible as a function of the depth, h, alone. This dimension, h, is then the only unknown quantity in the equation of stability, and can be, therefore, determined.

Whenever the unknown external load, which enters as a factor in the direct calculation of the moments and shearing forces, can be treated as part of the uniform load, and included under a definite functional form of the area of cross-section, the moment due to the uniform load thus made up of two distinct parts, may be substituted in the equation of stability, which will then contain only two unknowns; viz., the depth, h, and breadth, b, of the cross-section. In order to arrive at the definite values of, h and b, some ratio must be assumed to exist between these two linear dimensions of the beam.

In the case of a circular cross-section, the area, Ω, is a function of one dimension only; viz., the radius; for, then,

$$\Omega = \pi . r^2; \; I = \frac{\pi . r^4}{4}; \; r = \frac{h}{2}.$$

Consequently,

$$m = \frac{I}{\Omega} \div \left(\frac{h}{2}\right)^2 = \frac{1}{4};$$

and, therefore, by the general equation of stability [see above],

$$\frac{p}{\pi\, r^2} + p_o = \frac{1}{4} . \frac{2\,r}{a^2} . f,$$

or,

$$p + \pi p_o . r^2 = \frac{1}{2} . \frac{\pi r^3}{a^2} . f,$$

whence, multiplying both sides by, $\frac{2a^2}{\pi f}$,

$$\frac{2a^2}{\pi f} . p + \frac{2a^2}{f} . p_o . r^2 = r^3.$$

By transposition,

$$r^3 - \frac{2a^2}{f} . p_o r^2 - \frac{2a^2 . p}{\pi f} = o.$$

Taking a concrete example, let, $f = 4$ tons per square inch; $a = 30$ feet; $p = \frac{1}{5}$th ton per lineal foot; $p_o =$ weight of a prism of beam, one foot long and one inch square at base = (say), 0·002 ton. Then, substituting these data in the general equation, we find,

$$r^3 - \frac{9}{1440} . r^2 - \frac{50}{144\pi} = o,$$

to solve which, put

$$r = z + \frac{9}{1440 \times 3} = z + \frac{3}{1440} .$$

Hence, by substitution,

$$\left(z + \frac{3}{1440}\right)^3 - \frac{9}{1440}\left(z + \frac{3}{1440}\right)^2 - \frac{50}{144\pi} = o,$$

or,

$$z^3 - \frac{27}{100.(144)^2} z - \left(\frac{54}{1000.(144)^3} + \frac{50}{144\pi}\right) = o. \quad (3)$$

Let, moreover,

$$z = x + y,$$

then,

$$\begin{aligned} z^3 &= x^3 + 3x^2 y + 3y^2 x + y^3 \\ &= x^3 + y^3 + 3xy(x + y) \\ &= x^3 + y^3 + 3xy.z. \end{aligned}$$

Transposing,

$$z^3 - 3xy.z - (x^3 + y^3) = o.$$

In order that this last equation may be identical with equation, [3, p. 353], it is necessary and sufficient that,

$$3xy = \frac{27}{100.(144)^2}, \text{ and, } x^3 + y^3 = \left(\frac{54}{1000(144)^3} + \frac{50}{144\pi}\right),$$

by which relations,

$$xy = \frac{9}{100(144)^2}; \text{ and, } x^3 + y^3 = q \text{ (say).}$$

Here we have two equations to determine the two unknown quantities, x and y.

Raising the first to the cube power,

$$x^3 y^3 = \left(\frac{9}{100(144)^2}\right)^3,$$

but by the second,

$$x^3 + y^3 = q;$$

so that the terms, x^3, and, y^3, are connected by the quadratic equation,

$$t^2 - qt + \left(\frac{9}{100(144)^2}\right)^3 = o;$$

for, if, x^3 and y^3, be the two roots of this equation, the theory of equations furnishes the relations,

$$x^3 + y^3 = q; \; x^3 y^3 = \left(\frac{9}{100(144)^2}\right)^3.$$

Solving this reduced quadratic in, t, we find,

$$t = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} - \left(\frac{9}{100(144)^2}\right)^3}.$$

Hence the two roots are,

$$x^3 = \frac{q}{2} + \sqrt{\frac{q^2}{4} - \left(\frac{9}{100(144)^2}\right)^3}; \; y^3 = \frac{q}{2} - \sqrt{\frac{q^2}{4} - \left(\frac{9}{100(144)^2}\right)^3}$$

and therefore,

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \left(\frac{9}{100(144)^2}\right)^3}}$$

$$y = \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} - \left(\frac{9}{100(144)^2}\right)^3}}$$

But,

$$z = x + y\,;\ r = z + \frac{3}{1440},$$

wherefore,

$$r = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \left(\frac{9}{100(144)^2}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} - \left(\frac{9}{100(144)^2}\right)^3}} + \frac{3}{1440}.$$

The term, $\left(\frac{9}{100(144)^2}\right)^3$, may be neglected in comparison with, $\frac{q^2}{4}$, the term, q, being approximately equal to, $\frac{1}{9}$, so that,

$$r = \sqrt{q} + \frac{3}{1440}\,;$$

or, since $q = \frac{1}{9}$, very approximately,

$$r = \sqrt[3]{q} + \frac{3}{1440} = \sqrt[3]{\frac{1}{9}} + \frac{3}{1440}$$

$$= \tfrac{1}{2} \text{ foot, nearly,}$$

whence,

$$h = 2r = 12 \text{ inches.}$$

The quantities, x and y, used as auxiliaries in the preceding calculation, have each three values corresponding to the factors,

$$\sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \left(\frac{9}{100(144)^2}\right)^3}}\,;\ \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} - \left(\frac{9}{100(144)^2}\right)^3}}\,;$$

each multiplied by the three cubic roots of unity; namely

$$1\,;\ a = \frac{-1 + \sqrt{-3}}{2}\,;\ \text{and } a^2 = \frac{-1 - \sqrt{-3}}{2}$$

there being thus three values of, x, and three of, y; the sum $x + y$, can be varied in nine ways. But of these values of, x and y, only such are admissible as satisfy the equation,

$$x\,y = \frac{9}{100\,(144)^2}$$

In the present example the given real value of, r, corresponds to the root, $\sqrt[3]{1} = 1$.

From the preceding investigation it may be inferred generally that, whenever the area of cross-section, Ω, and its moment of inertia I, can be expressed in terms of *one* of the linear dimensions of the section, such as, r, the general equation of stability suffices of itself to determine the proper proportions of the cross-sections.

If, in the case just treated, the weight of the beam itself had been neglected, we should have obtained,

$$\frac{M_o}{I} \cdot \frac{h}{2} = \frac{\frac{1}{8}\,p\,a^2 . r}{\frac{\pi . r^4}{4}} = 144\,f = 576 \text{ tons,}$$

or,

$$\pi . r^3 = \frac{1}{288} . p\,a^2$$

$$= \frac{1}{288} \cdot \left(\frac{1}{9}\right)^{\text{ton}} \cdot (30)^2$$

$$= \frac{25}{72}$$

So that

$$r = \sqrt[3]{\frac{25}{72\pi}} = 0{\cdot}48 \text{ ft.}$$

$$h = 2\,r = 11{\cdot}52 \text{ ins.}$$

Therefore the difference between the diameters of beam, respectively including and excluding the weight of the beam itself, is,

$$d = 12'' - 11{\cdot}52'' = 0{\cdot}48'';$$

or a little less than half an inch.

Since the greatest longitudinal stress developed in a section of a girder, subject to load, is expressed in the form,

$$\frac{t}{\omega} = \frac{M}{I} \cdot \frac{h}{2},$$

where, h, is the depth of the girder or beam; it follows, that, if, M, remain constant at any particular section, the stress induced at that section will be in direct proportion to the factor, $\frac{h}{I}$; or inversely as $\frac{I}{h}$. To reduce the stress, it is well, therefore, to augment as far as possible the value of, $\frac{I}{h}$. Now, this ratio varies with the character of cross-section. For a section of rectangular form, we have,

$$\frac{I}{h} = \frac{\frac{1}{12} \cdot b\,h^3}{h} = \Omega.\,\frac{h}{12}.$$

For a section of elliptic form, the vertical axis being, h, and the horizontal one, b,

$$\frac{I}{h} = \frac{\frac{1}{64} \cdot \pi\,b.\,h^3}{h} = \frac{1}{64} \cdot \pi\,b\,h^2;$$

or, since, $\Omega = \frac{\pi}{4} \cdot b\,h$,

$$\frac{I}{h} = \frac{\pi\,b\,h}{4} \times \frac{h}{16} = \Omega.\,\frac{h}{16}.$$

Thirdly, for a section of the form of a double T, the material being supposed to be mostly condensed in the flanges, and the radius of gyration approximately equal to, $\frac{h}{2} = r_1$

$$\frac{I}{h} = \frac{\Omega\,.\,r_1^2}{h} = \Omega\,.\,\frac{h}{4}.$$

Comparing the above three values of, $\frac{I}{h}$, it will be seen that for the same area of cross-section, Ω, the value of the

ratio, $\frac{I}{h}$, is greatest in the case of a double, *T*, section, least for an elliptic section, and a mean for a cross-section of rectangular form. Wherefore the best form, as far as resistance to bending moment is concerned, is the double, *T*.

8. DEFLECTION AND FRACTURE OF BEAMS.—If a beam or girder of length, $2a$, and supported at both ends, sustain a load, P, concentrated at the centre of span, the maximum central ordinate of deflection is expressed by,

$$y_0 = \frac{P \cdot a^3}{6 \cdot E I} \cdot$$

If the section be rectangular of a depth, h, and breadth, b, this value of, y_0, becomes,

$$y_0 = \frac{P\ a^3}{6 E \cdot \frac{b h^3}{12}} = \frac{2 \cdot P a^3}{E \cdot b h^3};$$

so that the central deflection, which is also the maximum deflection, of a beam composed of homogeneous metal, for which, E, is constant, varies directly as the first power of the load and the third power of the span; inversely, however, as the first power of the breadth, b, and the third power of the depth, h, of cross-section.

In a beam, having a rectangular form of cross-section, of depth, h, and breadth, b, the stress per unit of cross-section produced by a load, P, applied at the centre of span and a uniformly distributed load, p, where, p, may be taken to represent the weight of the beam per unit of length, will be given by the formula,

$$\frac{t}{\omega} = \frac{M}{I} \cdot \frac{h}{2} = \frac{\left[\frac{P}{2} a + \frac{1}{2} p a^2\right] \cdot \frac{h}{2}}{\frac{1}{12} \cdot b h^3}$$

$$= \frac{3 a (P + p a)}{b h^2}.$$

If, therefore the weight, P, be increased gradually in amount

till the beam gives way under a load-limit, P_o, the stress corresponding to this maximum load; namely,

$$\frac{t_o}{\omega} = \frac{3\,a\,(P_o + p\,a)}{b\,h^2},$$

has been called the *breaking strain* of the material employed.

Rigorously speaking the term "*breaking strain*" is not a very appropriate expression; since the equation of stability,

$$\frac{t}{\omega} = \frac{M}{I} \cdot \frac{h}{2},$$

has been deduced on the supposition that the elongation under increased loads continues proportionate to the pressure applied. At and before rupture these elongations cease to bear a uniform ratio to the applied loads;—hence the gap in the argument used in support of the term "*breaking strain.*"

EXAMPLES.

1. Assuming a solid girder, $A\,B$, 20 feet long, to be loaded at a point, C, distant 4·4 feet from, A, with a weight of 5 tons; find the reaction at the abutment, A.

Reaction = 3·9 tons.

2. In the same example find the bending moment at a vertical section through the centre of weight, C.

Bending Moment = 17·2 foot-tons.

3. Find the bending moment at the centre of span of the same girder, due to the load of 5 tons at, C, where as before, $A\,C = 4{\cdot}4$ feet.

Bending Moment = 11 foot-tons.

4. Taking a solid girder, 20 feet span, loaded at the centre with a weight of 5 tons; find the bending moment at the central section.

Bending Moment = 672,000 inch-lbs.

5. Assuming the cross-section of the same girder to be of a double, T, form, and of the dimensions :—total depth, $h = 24$ inches ; breadth of flanges, $b = 12$ inches ; breadth of web = 1 inch ; and supposing, moreover, the chief part of the material to be concentrated in the flanges at points distant, $\frac{h}{2} = 12$ inches, from the neutral axis ;—find the Moment of Inertia of the section, neglecting the web.

Moment of Inertia = $10{,}36\dot{8}$.

6. Find the unit-stress, induced at the central section, corresponding to the given load of 5 tons, and the Moment of Inertia just determined.

Unit-Stress = 778 lbs. per sq. inch.

7. Find the central deflection of the same girder under a load of 5 tons at the centre of span.

Central Deflection = $0{\cdot}008\dot{8}$ inch.

8. What would be the uniform load per linear foot, producing the same central deflection ?

Uniform Load = $\frac{2}{5}$th ton per linear foot.

9. What would be the uniform load per linear foot, producing the same bending moment at the centre of span as the concentrated central load of 5 tons ?

Uniform Load = $\frac{1}{2}$ ton per linear foot.

10. Lastly, assuming the bending moment at the centre to be due to a concentrated load of 5 tons, shew that the equation, adopted by M. Kleitz for ordinary bridges, furnishes a value for, p, of 1,120 lbs. per linear foot.

CHAPTER VIII.

WIND PRESSURES.

1. GENERAL THEORY.—It is generally admitted that the pressures of wind-currents vary as the squares of their velocities. This relation is a deduction from a mass of meteorological data, collected from time to time. Whilst, however, it is almost a certain fact that wind-pressures and wind-velocities are related in this way, much uncertainty surrounds the values to be attributed to the coefficients, connecting these two factors in one formula.

Most of the experiments on wind-pressure, hitherto made, have been carried out by the aid of test-plates, and the pressures registered by means of the resistance of springs, attached to the plates. This method is, to say the least, very unsatisfactory and unscientific. Springs introduce the mystical factor of impact, and always give values considerably below the actual pressure of the wind. This matter is worthy of mention; for, even at the present time, we find unpractical suggestions made by civil engineers, who have a large practice in bridge-building; and, therefore, have to deal with this important but much neglected question of wind-pressure. In the discussion on wind-pressures, held at the Institution of Civil Engineers, in March 1882, Mr. B. Baker recommended the experiment of "slinging the idle girders of the Tay Bridge, with springs at the side, so as to measure the absolute pressures of the wind." Now, it is perfectly certain that a mighty girder of this kind would not register small pressures at all; insomuch that the force of moderate winds would be insufficient to overcome

the inertia of the girder. On the other hand, it is equally certain that, owing to the same cause and to the short duration of gusts or maximum wind velocities, this gigantic anemometer would register comparatively low pressures during high winds. Moreover, the evil influence of impact would require to be taken into account, before accurate results could be obtained.

No doubt, in proposing the above experiment, Mr. Baker wished to realise as nearly as possible the actual conditions of a girder, submitted to the action of wind. But this object cannot be perfectly attained, unless the reactions at the ends of the girder be made absolutely rigid; and, even then, granting that the resultant pressures could be approximately ascertained, a large girder of this kind could never serve the purpose of an accurate anemometer. The measurement of wind-pressure is an operation requiring great care and skill, attended by instruments of delicate manufacture, small size, weight, and inertia.

Although it is not an easy matter to measure accurately the pressure of the wind, there can be no doubt that air-currents, striking large and exposed surfaces, create enormous pressures, which are often accidentally increased by other causes. A sudden gust of wind, driven against a long-span railway bridge, will be divided into two currents at the surface of the opposing girder. Part will pass over and part under the bridge. The under current will probably curl round the platform in whirling eddies, and, if the bottom of the floor be formed into a number of cells or open boxes, by a system of stringers and projecting beams, the volume of the wind will rush into these cavities, from which it will find no escape, save in the line or direction by which it entered. In other terms, it has but one line of discharge. An isolated test-plate has four edges; and, consequently, four lines of discharge. When, as in the case of honey-combed surfaces, three of these lines are cut off, enormous pressures are accidentally developed by reason of the *accumulated head* of wind, arising from slow and inadequate discharge.

Supposing, therefore, the four lines of discharge of the test-plate to be reduced to one, it would seem no exaggeration to

add *one-fourth* of the registered test-plate pressure, for every line of discharge cut-off. According to this rule, a test-plate pressure of 30 lbs. per sq. ft. would be increased by *three-fourths* to 52·5 lbs. per sq. ft., when impinging upon a cellular surface. This is equivalent to assuming the coefficient of discharge of honey-combed surfaces to be, 1¾.

In the present state of uncertainty, which surrounds the subject, and viewing the fact that wind, striking an uneven surface, may be greatly impeded in its discharge, it seems scarcely justifiable to design irregular or honey-combed surfaces for a wind-pressure of less than 56 lbs. per sq. ft. of exposed section, which is the limit adopted by the Board of Trade. But this high limit might be safely reduced in favour of structures, presenting few cavities for the accumulation of wind-pressures.

In order to resist the effects of wind, the tops and bottoms of bridge-girders are often bound together by a system of horizontal bracing forming, as it were, a pair of horizontal girders, connected by the plates of the ordinary girders. These horizontal girders, when subjected to the uniform force of the wind, distributed over the opposing plate-surface of the vertical girder, can be treated in the same way and by similar methods, as ordinary vertical lattice girders supporting vertical loads. In the former case, we have to deal with two horizontal girders, separated by the distance between the top and bottom booms; in the latter instance, with two vertical girders separated by the width of the bridge.

In the treatment of vertical girders, subject to vertical loads, the first step is to determine the reactions at the tops of the neighbouring piers; and, then, by a subsequent calculation or graphic construction, to find the stresses induced by the given loads in the bars of the trellis-work.

In a similar way, when treating the horizontal girder-system above described, we must first find the reactions at the tops of the neighbouring piers, due to a uniform wind-load distributed over the joints of the horizontal girder. The reaction at each pier of a single independent span is equal to *half* the wind-load, bearing on the span. This half wind-

load will induce two kinds of reactions, one at the top of the pier, which is connected with the system of horizontal bracing; another at the base of the same pier, which, combined with the half wind-load at the top, will give rise to a bending moment, similar to that brought to bear upon a horizontal beam, fixed at one end and free at the other, and acted upon, at its free extremity, by a vertical force. Imagine the horizontal beam to become vertical and the vertical force horizontal, and the analogy is complete. The bending moment, due to wind, will tend to upset the pier over its leeward edge.

To counteract this effect, we have the moments of stability of the weight of the pier, of the superincumbent dead-weight of the span, and of the live load, which may happen to be passing over the bridge. Should these weights prove sufficient of themselves, to counterbalance the upsetting action due to wind-pressure, no anchorage-bolts, fastening the bottom of the pier to the foundation, are required. But, in some cases, when the combined weights of the pier and superstructure are insufficient, it becomes necessary to add anchorage-bolts, exerting specified resistances. Further to strengthen the pier, it may be found convenient to add a transverse system of bracing between the half wind-load at the top, and the corresponding reaction at the base of the pier. This can be done in the way exemplified in the "Bouble" viaduct, which example has been very elegantly worked out by M. Gaudard, professor of civil engineering at the Academy of Lausanne, in a paper presented to the Institution of Civil Engineers.* We shall return later to this interesting example, and treat it in detail.

2. CROSS-BRACING.—Given a system of cross-bracing, such as that shewn in Fig. 185, Pl. III., the platform being subject to two kinds of loads; namely, the downward weight, W, including the live and dead loads upon this part of the bridge; and the inclined load, R, representing the wind-pressures, equably distributed over the under-surface of the platform; —to find the stresses in the braces, CE, CF; DE, DF.

It will be found convenient to treat the forces, W and R,

* Minutes of Proceedings, Vol. LXIX. Part iii.

separately, and then to combine the component stresses, due to these separate loads, so as to find the resultant stress in any of the bars.

If we wish to follow a rigorous method, it is necessary first to find the reactions at A and B due to W; and to treat the span, $A B$, as a continuous girder of three spans, supported at the two intermediate points, C and D. Thus we should find the shearing forces at, C and D, and resolve them respectively along, $C E$ and $C F$; $D E$ and $D F$. In this way we should determine the compressions, due to W, in the four members of the cross-bracing.

Next, it would be necessary to resolve, R, into a vertical force R_1, and a horizontal force, R_2. The force, R_1, will determine tensions in the bars, $C E$, $C F$, $D E$, $D F$, which can be found by a method perfectly similar to that just explained for finding the stresses due to W. The remaining force, R_2, will constitute a thrust, acting upon the platform and tending to shift it to the left. It will produce tensions in the bars, $C F$ and $D F$, and compressions in the bars, $C E$ and $D E$. The amount of these tensions and compressions can be determined by the following rule.—The shearing forces at, C and D are known. Call them, C_s and D_s; and construct a right-angled triangle, Fig. 186, making, $\overline{A B} = R$; and, therefore, $\overline{A C} = R_1$; $\overline{B C} = R_2$. Make, $\overline{A C_s} = C_s$; $\overline{A D_s} = D_s$; and draw $\overline{C_s K}$ and $\overline{D_s L}$, parallel to $\overline{C B}$. Then, $\overline{C_s K}$, will give the horizontal thrust at C, according to the scale by which $\overline{B C}$, represents, R_2. Similarly, $\overline{D_s L}$ represents the thrust at D. Let, therefore, a^o, be the angular inclination to the vertical of the direction of wind-pressures; then,

$$\frac{C_s K}{A C_s} = tan.\ a\ ;\ \overline{C_s K} = C_s\, tan.\ a = H_c$$

and,

$$\frac{D_s L}{A D_s} = tan.\ a\ ;\ \overline{D_s L} = D_s\, tan.\ a = H_d.$$

The thrusts, H_c and H_d, can then be resolved along the bars meeting at, C and D, by the principle of the triangle of forces. Thus, at C, the local horizontal thrust will produce a definite

tension in bar, CF, and a proportionate compression in bar, CE. The resultant stress in any of the bars, CE, CF, DE, DF, will be represented by the algebraical sum of the tensions and compressions, due to the separate loads, R_1, R_2, and W, thrusts being represented by a *plus* and tensions by a *minus* sign.

The above method is rigorous, but not generally followed in practice. For practical purposes it is considered sufficient to take half the loads lying between, CA and CD, for the resultant load acting at, C. Similarly, the resultant load at, D, is equal to half the loads upon, DC and DB. These resultant loads can then be resolved along, CE, CF; DE, DF, according to the principle of the parallelogram of forces.

3. ELEVATED WIND-CURRENTS.—In dealing with the influence of elevation upon wind-pressures, I cannot do better than summarise the observations of Mr. Rogers Field, published in the Minutes of the Proceedings of the Institution of Civil Engineers, Vol. LXIX. Session 1881-82. Mr. Field observed that it was clear from the form of the curve, Fig 187, even regarding it as only a very general indication of what took place, that no pressure could be assigned which was suitable for all elevations. Assuming a pressure of 56 lbs. per square foot as correct for an elevation of 200 feet, there would only be a pressure of about 20 lbs. at a height of 15 feet. Hence, wind-pressures would vary with the heights of structures.

Another important point had reference to Robinson's cup-anemometer. The pressures made public were frequently not directly measured, but deduced from velocity-measurements, and it was therefore of great importance to know how those measurements were taken, and how far they were reliable. In the standard instrument (Kew pattern), there was a distance of 4 feet from centre to centre of cups, and the cups themselves were 9 inches in diameter. The cups revolved at a less velocity than the wind, and the rate was multiplied by a certain factor to find the velocity. It might appear to be rather a startling statement, but it was strictly true, that nearly every observation by that instrument of a high velocity which had been quoted in England, was about 20 per cent. too high.

The instrument was invented more than thirty years ago by Dr. Robinson, who stated that the wind moved at thrice the velocity with which the centre of the cups moved; so that, if the cups travelled one mile, the wind went three miles. That rule was adopted by all the instrument makers, who made a dial at the foot of the instrument, registering thrice the velocity of the cups, and called the registration of this dial the velocity of the wind. At all the Observatories, and notably at Kew, where the anemometers were sent to be tested, the calculations were made on that assumption; but it was now admitted by almost all, who had studied the subject, that the multiple of *three* was about 20 per cent. too high. According to recent experiments a velocity of 15 miles per hour of the cups represented about 36 miles per hour velocity of the wind, instead of 45 miles per hour as assigned by Dr. Robinson's rule. If 36 be divided by 15, it gave a coefficient of 2·4 instead of 3, shewing that the factor, 3, was about 20 per cent. too high.

4. RATIO BETWEEN VELOCITY AND PRESSURE.—The pressure of a volume of air in motion, striking a plane perpendicularly, and escaping by rebound, is theoretically equal to the weight of a column of air of *twice* the height corresponding to its velocity. Thus, let

v = the velocity of the air in feet per second.
h = the height through which a body must fall to produce the velocity, v.
w = the weight in lbs. per cubic foot of air.
g = the coefficient of gravity.
p = the wind-pressure in lbs. per sq. ft.

Then,

$$h = \frac{v^2}{2g}; \; p = \frac{w.v^2}{g},$$

or, according to Mr. T. Hawksley, for atmospheric air,

$$p = \frac{0{\cdot}0765.\,v^2}{32}$$

$$= \left(\frac{v}{20}\right)^2, \text{ very nearly.}$$

Adopting this formula, Mr. T. Hawksley has calculated the series of pressures, contained in the following Table :—

Velocities in Feet per second.	Velocities in Miles per hour.	Pressures in lbs. per square foot.
10	6·8	0·25
20	13·6	1·00
30	20·4	2·25
40	27·2	4·00
50	34·0	6·25
60	40·8	9·00
70	47·6	12·25
80	54·4	16·00
90	61·2	20·25
100	68·0	25·00
110	74·8	30·25
120	81·6	36·00
130	88·4	42·25
140	95·2	49·00
150	102·0	56·25

These values represent maximum normal pressures. In cases where the wind strikes any object, such as the roof of a building, at an angle, θ, the pressure would be diminished in the ratio of the square of the sine of this angle; or

$$p = \left(\frac{v \sin. \theta}{20}\right)^2$$

5. Wind-Action upon Roofs.—In, Fig. 188, the wind is supposed to be blowing, in a downward direction, upon the right-hand side-rafter of the roof, and to be proportionately distributed over the joints, $h\,i$, $i\,j$, and $j\,k$. At each joint the two forces, due to dead weight and wind-load, are combined in one resultant force. The reactions $f\,e$ and $k\,e$, are supposed to act parallel to the resultant of the system, the path, magnitude, and direction of which are found by the aid of the polar polygon and the polygon of forces, both of which are shewn upon the figure. This and the next example are treated by the method of lettering explained in Pt. I. Ch. II.

6. The Effect of Wind upon Braced Piers.—In Fig. 190, the wind is assumed to be blowing normally against the face of the pier, tending to overturn it on its leeward edge. It has been shewn that wind-pressures increase with elevation,

(§. 3), and the wind-load is distributed over the joints, partially in accordance with this principle. A wind-load of 20 tons is concentrated at the top joint, $k\,j$; a wind-load of 10 tons at $k\,l$; an equal load of 10 tons at, $l\,m$; and a wind-load of 5 tons at the base of the pier.

These wind-loads are then composed with the partial dead weights or superincumbent loads acting at the respective joints; and the resultants, so found, form a series of inclined forces applied at the same joints. Thus, for example, the superincumbent weight of 20 tons, and the indicated wind-load of equal amount, acting at the summit of the pier, are both combined in the single resultant force, *F*, applied at the top joint.

The loads acting on the right of the pier are those due to the weights of the parts, and any external forces brought to bear upon special joints. The actual loads are 4 tons at the joint, $j\,i$, 5 tons at, $h\,i$, and 13 tons at, $h\,g$.

The polygon of known forces, $n\,m\,l\,k\,j\,i\,h\,g$, Fig. 191, is then constructed; and, subsequently, the corresponding polar polygon, relatively to the pole, *O*, which determines a point on the objective path of the resultant force. The moment of this resultant about the leeward edge of the pier can be found by any of the methods explained in Chapter V., and thence the necessary tension of anchorage is easily deduced; and the polygon of forces completed, by drawing a line, $n\,f$, parallel and equal to the tension of anchorage, and a line, $g\,f$, representing the magnitude and direction of the resultant reaction at the leeward edge of the pier. The rest of the reciprocal figure is drawn in the usual way.

The Bouble Viaduct.—In dealing with this example, I shall take substantially the same treatment described by Professor Gaudard, in the *Minutes of the Proceedings of the Institution of Civil Engineers, Vol. LXIX.*

Having ascertained the lateral force, exerted by the wind against the pier, it is necessary to calculate the special molecular strain which it tends to set up, in order to add it to that produced by the permanent and moving loads. In resisting the wind, the roadway acts as an imaginary girder whose

flanges are the actual girders of the bridge, and whose lattices are the horizontal braces and wind-ties. Moreover, the resistance offered by the irregular interlacing motion of the trains must be taken into account. Owing also to the wind coming in gusts, its effect on each girder, whether tensive or compressive, must be considered as added to the strain due to load, and in the case of several spans the most unfavourable condition must be allowed for.

An arch has the advantage over a straight girder of opposing less surface to the wind in the central portion, whilst the opposite is the case with a bow-string.

Two examples of iron arches, with narrow roadways, spanning very large openings, are those of Oporto, on the Douro, which has a width of 14 feet 9 inches between the parapets, and a span of 525 feet, and that of Montereale, over the Cellina torrent, which has a width of 9 feet 10 inches and a span of 272 feet. Both these bridges are secured against the wind by special contrivances; the first, by giving a batter of 0·1164 to each face of the bridge, so that the distance from centre to centre of the arched ribs, which is only 12 feet 10 inches at the crown, is increased to 49 feet 2½ inches at the springings; the second by an external wind-bracing; namely by side-buttresses coming from the haunches of the arch, and butting against the masonry at two points 27 feet 7 inches apart; whereas the distance between the arched ribs is only 9 feet 10 inches.

Certain structures may be liable to be wholly overturned by gusts of wind. Iron superstructures are generally free from this danger in consequence of their weight, except perhaps during the time of erection. On the contrary, the iron piers of high viaducts need to be firmly anchored to their masonry pedestals, as M. Nordling has pointed out in his memoir about various works on the branch lines of the Orleans Company*. These kinds of piers are eventually strained as elastic braced structures fastened at their base and subjected at their summit to violent horizontal thrusts.

* Annales des Ponts et Chaussées, 1864 and 1870.

On this account, instead of distributing their mass in a number of external and internal uprights, it is better to concentrate it at the angles in only four ribs connected together by cross-bracing. The anchorage at the base is rendered more effective, by fastening buttresses to the piers, so as to enlarge their bases. If the height does not exceed 130 feet, as for instance, in the Bellon viaduct, the uprights may be curved outwards, so as to widen the base, without the aid of special stays. One of the high piers of the Bouble viaduct, Fig. 192, will serve as an example to illustrate, by an approximate process, to what severe strains such a structure might occasionally be exposed. M. Nordling has assumed the wind-pressure at 55·3 lbs. per square foot, without allowing for a train on the bridge; as, in his opinion, if such a storm ever burst upon this structure, the traffic would be suspended for a time; and, moreover, the above pressure appears to him excessive for the locality. Let, however, the worst possible case be considered by imagining a concurrence of adverse circumstances, the structure being in a very exposed situation, and the full fury of the gale suddenly occurring whilst a train is passing over.

Taking only a half pier containing two uprights and the intermediate bracing, the span being 164 feet, crossed by two lattice girders 14 feet 9 inches high, it appears that, allowing for the spaces, the wind, having a pressure of 55·3 lbs. would exert a stress of about 20 tons at a height of 196·2 feet above the footings, which gives a moment of 3,924. The pressure on the train is 16·2 tons, with a leverage of 210·3 feet, giving a moment of 3,407. Lastly, the moment of the pressure of the wind on the half pier amounts to 20 tons $\times$ 92·85 feet $=$ 1,857. Thus the total moment, or turning power, on the leeward edge of the base is 9,188. The moment of stability, due to the loads, is obtained as follows: taking 60 tons as the weight of the half span, and 120 tons as the weight of the half pier, (the cast-iron cylinder being ballasted with concrete), and allowing 42·5 tons as the weight of the train which suffices to prevent its being overturned by the gale, the total weight amounts to 222½ tons, and the half width of the base

being 33·8 feet, the moment is 7,520, leaving a deficiency of 1,668. To provide for this, the anchorage-tie must exert a tension of $\frac{1,668}{67 \cdot 6} = 24 \cdot 69$ tons. Without the help of the buttresses, the width of the base of the pier would be only 24 feet 3 inches, instead of 67 feet 7 inches, and the anchorage would be subjected to the great strain of 267 tons. The reciprocal diagram of the half pier, considered as an articulated structure, is given in, Fig. 193, from which a notion may be formed of the enormous pressures arising under heavy winds. The lattice is hypothetically reduced to the lines of Fig. 192, by omitting as well the foot of the straight uprights, replaced by the corresponding curved or polygonal stay; as, in each row of cross-bracing, that diagonal which, exposed to a wind from the left, would be strained in compression, and is thought too flexible to offer an effectual resistance in this way.

The external forces, applied to the various summits, produce the following horizontal components. At the summit, *A*, the whole force of the wind against the beams and the train is brought to bear; namely, a force of 40·04 tons, obtained by dividing the moment, 7,331, by the height, 183 feet, of the point, *A*, above the base. The pressure against the half pier amounts to about 2 tons acting at each of the joints, *B*, *G*, *H*, . . . *I*, facing the wind. The weights or vertical forces are ;—51·25 tons at *A*, due to the loaded roadway; the same weight at, *B*, increased by a portion of the pier, amounting altogether to 57·25 tons; lastly, at each of the points, *G*, *H*, . . . *I*; and, *C*, *D*, . . . *E*, a vertical force of 6 tons. The reactions in equilibrium, developed by the base of the support, are at *K*, the tension of anchorage, amounting to 24·69 tons as calculated above, and acting from top to bottom; at *F*, a vertical upward reaction equal to the total weight increased by the strain of anchorage; namely, to 247·2 tons; and a horizontal force acting from right to left, which, counteracting in projection all the wind-pressures, is equal to 60·04 tons. Consequently, the resultants at the different points assume oblique or vertical directions. The

oblique resultants are ; 65·04 tons at *A* ; 6·3 tons at each of the joints, *G*, *H*, . . . *I*, of the left upright ; and 254·4 tons at the leeward edge, *F*. The state of equilibrium of the external forces is shewn by the closed polygon in double lines, Fig. 193. The reciprocal figure is completed by the addition or grouping of a series of other closed polygons, representing the respective states of equilibrium of the various summits of the articulated system, Fig. 192, under the influence of the internal and external forces, acting on each of them. The inscription of identical numbers in Figs, 192, 193, serves to indicate their connexion ; thus, for example, the closed polygon, 8, 9, 11, 12, 6·3 tons in Fig. 193, proves that the point, *H*, of Fig. 192 is in equilibrium under the influences of the external force, 6·3 tons ; the tensional strains of the bars, 8, 9, 12 ; and the compression of bar, 11. It will be observed that the left side is in tension from, *G* to *K*, the greatest tensional strain of about 190 tons, occurring in bar, 34. With a cast iron pipe having an external diameter of 1 foot 8 inches, and an internal diameter of 1 foot 4 inches, this strain would amount to 1·9 tons per square inch ; but, as previously stated, the Bouble viaduct was constructed on the supposition of the maximum pressure being less. The compressive strain reaches 422 tons in bar, 40, which would amount to 4·1 tons per square inch ; but in reality the strain is less if the uprights are made complete, as in the true section of the pier. Moreover, it is certain that the rigidity of the cast iron columns and their bolted flange-joints must considerably modify the conditions of the problem. Instead, therefore, of merely comparing the pier to an articulated system, each member of which is considered to be free to deflect in any way, as assumed above, it would be necessary, in a complete design, to study the transmission of force resulting from impeded deflection.

We have not much to add to the above lucid exposition of the graphic treatment of the stresses induced in the half pier of the Bouble viaduct. M. Gaudard wisely considers the load at each joint to be made up of half the loads, horizontal or vertical, applied between the neighbouring joint-spans. He is

also right in considering the resultant reaction of the system to pass through the leeward edge, *F*, of the pier; since, when the pier is about to upset, all forces are concentrated at this point. It is also correct to look upon all wind-forces and statical loads as being finally transmitted to and resisted at the foot of the pier. Such will always be the case in a system of cross-bracing of this kind.

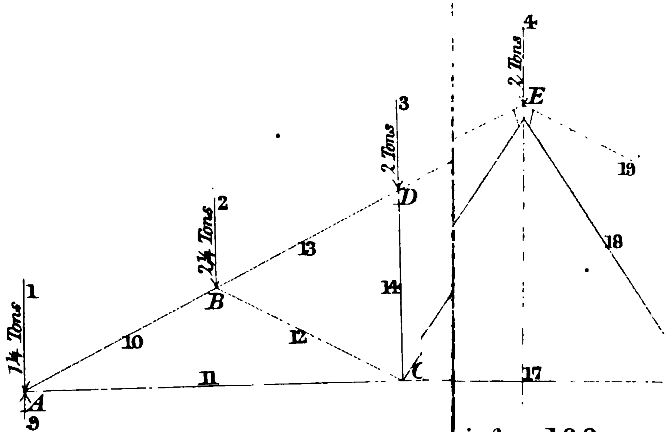
But, when treating cross-bracing of the horizontal type, lying in a plane parallel to the floor of the bridge; the reaction must be considered as occurring primarily at the pier-top, and then by translation at the foundation of the pier. A moment is introduced by the transference. and this moment actually exists.

If there be two systems of horizontal bracing, the wind load will be divided between them in a ratio determined by their position. If the two systems lie at equal distances on each side of the centre-line of wind-forces, they will each take half the total effect of wind. Supposing there be several systems of cross-bracing, occurring at intermediate points of the span, we should first find the shearing forces at these sections, due exclusively to wind-load, and apportion each of the shearing forces to its own system of transverse bracing.

If the horizontal system of bracing, belonging to one span, be not terminated at the ends of the span, but continued over several spans, the system must be treated by the theory of continuous girders, which is beyond the scope of this volume.

Fig. 165.

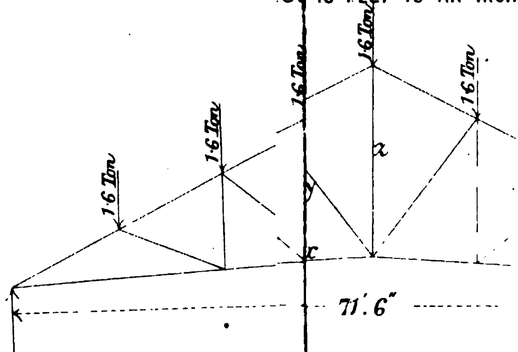
SWAIA STATION RC



ig. 166.

GODSHED AT WEYM

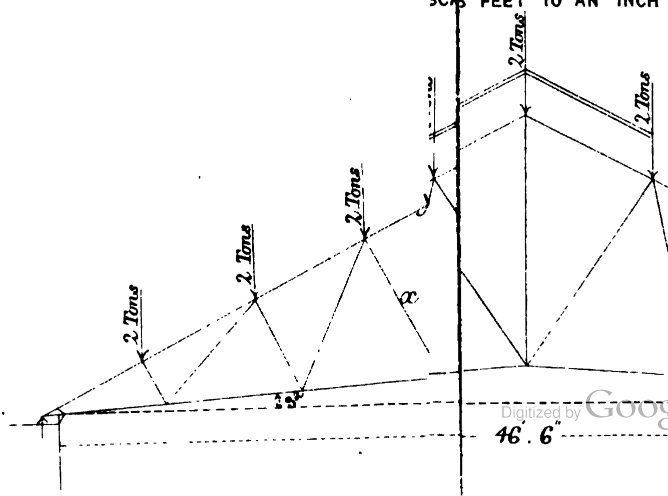
SC 16 FEET TO AN INCH



ig. 167.

RO F DIDCOT PROVENDE

SC FEET TO AN INCH.



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